DIRECT PROOFS OF THE FEIGIN–FUCHS CHARACTER FORMULA FOR UNITARY REPRESENTATIONS OF THE VIRASORO ALGEBRA

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ABSTRACT. *Previously we gave a proof of the Feigin–Fuchs character formula for the irreducible unitary discrete series of the Virasoro algebra with $0 < c < 1$. The proof showed directly that the mutliplicity space arising in the coset construction of Goddard, Kent and Olive was irreducible, using the elementary part of the unitarity criterion of Friedan, Qiu and Shenker, giving restrictions on h for $c = 1 - 6/m(m+1)$ with $m \geq 3$. In this paper we consider the same problem in the limiting case of the coset construction for $c = 1$. Using primary fields, we directly establish that the Virasoro algebra acts irreducibly on the multiplicity spaces of irreducible representations of $SU(2)$ in the two level one irreducible representations of the corresponding affine Kac–Moody algebra. This gives a direct proof that the only singular vectors in these representations are those given by Goldstone's formulas, which also play an important part in the proof. For this proof, the theory is developed from scratch in a self–contained semi–expository way. Using the Jantzen filtration and the Kac determinant formula, we give an additional independent proof for the case $c = 1$ which generalises to the case $0 < c < 1$, where it provides an alternative approach to that of Astashkevich.*

CONTENTS

1. Introduction. The Witt algebra $\mathfrak{witt}$ is the Lie algebra of complex vector fields $f(\theta) \cdot d/d\theta$ on the unit circle with $f(\theta)$ a trigonometric polynomial. It has a basis

$$\ell_n = ie^{in\theta} \frac{d}{d\theta}$$

for $n \in \mathbb{Z}$ and commutation relations

$$[\ell_m, \ell_n] = (m - n)\ell_{m+n}.$$

This Lie algebra has a central extension by $\mathbb{C}$, called the Virasoro algebra $\mathfrak{vir}$, with basis L_n $(n \in \mathbb{Z})$, C and the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} C.$$

In this article we will be interested in positive energy unitary representations of the Virasoro algebra. By this we mean an inner product space $\mathcal{H}$ on which the Lie algebra $\mathfrak{vir}$ acts by operators $\pi(L_n)$ satisfying $\pi(L_n)^* = \pi(L_{-n})$, $\pi(C) = \pi(C)^*$, with $\pi(C)$ acting as a scalar cI and such that $\mathcal{H}$ has a decomposition as an algebraic orthogonal direct sum of egenspaces of $\pi(L_0)$

$$\mathcal{H} = \oplus_{n \geq 0} \mathcal{H}(n)$$

with $\mathcal{H}(n)$ finite–dimensional and $\pi(L_0)\xi = (h+n)\xi$ for $\xi \in \mathcal{H}(n)$. Vectors in $\mathcal{H}(n)$ are said to have energy $h+n$. Necessarily h and c are real; and it is easy to see that $c \geq 0$ and $h \geq 0$. Every positive energy unitary representation can be written as a direct sum of irreducible representations which are uniquely determined by the lowest eigenvalue h of $\pi(L_0)$. In this case $\mathcal{H}(0)$ is one–dimensional and generated by a unit vector ξ_h. The character of a positive energy representation on $\mathcal{H}$ is defined as the formal sum

$$\chi_{\mathcal{H}}(q) = q^h \sum_{n\geq 0} d(n)\, q^n,$$

where $d(n) = \dim \mathcal{H}(n)$. We shall often write L_n in place of $\pi(L_n)$, regarding $\mathcal{H}$ as a module over the Virasoro algebra. The above character is then formally

$$\sum_{n\geq 0} \mathrm{Tr}_{\mathcal{H}(n)}\, q^{L_0} = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0}.$$

The work of Friedan–Qiu–Shenker **[12]** shows that the irreducible positive energy unitary representations are given precisely by a continuous series $c \geq 1$ and $h \geq 0$ together with a discrete series

$$c = 1 - \frac{6}{m(m+1)},\ h = h_{p,q}(m) \equiv \frac{(p(m+1)-qm)^2 - 1}{4m(m+1)},\ m \geq 2,\ 1 \leq q \leq p \leq m-1.$$

The case $m = 2$ corresponds to $c = 0$ and had already been analysed by Gomes: only the trivial one–dimensional representation can occur ($h = 0$). Otherwise the representations are all on infinite–dimensional inner product spaces $L(c,h)$. The problem of computing the character $\chi_{c,h}(q) = \chi_{\mathcal{H}}(q)$ when $\mathcal{H}$ is the irreducible representation with parameters (c,h) was first solved by Feigin and Fuchs **[8]**. We outline their proof and some of its subsequent simplifications below. It relies on a detailed knowledge of singular vectors and non–unitary representations.

The vector $\xi = \xi_h$ satisfies $L_n\xi = 0$ for $n > 0$ and $L_0\xi = h\xi$. The space $\mathcal{H}$ is therefore spanned by all monomials $L_{-k}^{n_k}\cdots L_{-1}^{n_1}\xi$ with $n_i \geq 0$. On the other hand there is a universal module for the Virasoro algebra, the Verma module $M(c,h)$, which has as basis such vectors. Formally it can be defined as the the module induced from the one–dimensional representation $L_n = 0$ ($n > 0$), $L_0 = h$ and $C = c$ of the Lie subalgebra spanned by L_n ($n \geq 0$) and C. Its existence is standard and follows from the Poincaré–Birkhoff–Witt theorem applied to the universal enveloping algebra of the Virasoro algebra. (By that result the universal enveloping algebra has a basis consisting of monomials of the above form post–multiplied by monomials $C^k L_0^{m_0} L_1^{m_1}\cdots$ with $k \geq 0$ and $n_i \geq 0$.) By definition $M = M(c,h)$ has positive energy and $d(n) = \dim M(n)$ is given by the number of partitions $\mathcal{P}(n)$ of n. Thus

$$\chi_{M(c,h)}(q) = q^h \prod_{n\geq 1}(1-q^n)^{-1}. \tag{1}$$

By the definition of $M(h,c)$, the irreducible representation $L(c,h)$ is the quotient of $M(c,h)$ by its unique maximal submodule.

The Verma module is in general not an inner product space, but for c,h real, it does carry a unique Hermitian form (ξ,η) such that $(\xi_h,\xi_h) = 1$ and $(L_n\xi,\eta) = (\xi, L_{-n}\eta)$, the so–called *Shapovalov form.* It is easy to see that the maximal submodule of $M(c,h)$ coincides with the kernel of the Shapovalov form. Thus for all real values of c and h, the Shapovalov form defines a non–degenerate Hermitian form on $\mathcal{H} = L(c,h)$. The representation is unitary precisely when this Hermitian form is positive–definite on $\mathcal{H}$, or equivalently, since the subspaces $\mathcal{H}(n)$ are orthogonal, on each $\mathcal{H}(n)$.

In **[36]** we gave a direct method of determining the characters of the discrete series $0 < c < 1$ involving only unitary representations, which we summarise below. When $c > 1$ the character formula is particularly simple because the Verma module is itself irreducible so the character formula is given by (1). This remains true in the limiting case $c = 1$ provided $h \neq m^2/4$ with $m \in \mathbb{Z}$. When $h = m^2/4$ or equivalently j^2 with j a non–negative half–integer, the Verma module is no longer irreducible and the character formula takes the form

$$\chi_{L(1,j^2)}(q) = (q^{j^2} - q^{(j+1)^2}) \prod_{n\geq 1}(1-q^n)^{-1}. \tag{2}$$

The main result of this article is a direct proof of this formula using only unitary representations; it is the counterpart of our previous proof for the discrete series and relies only on a detailed knowledge of the boson–fermion correspondence in conformal field theory.

Before explaining the proof for $c = 1$, it will be helpful to recall the method used for the discrete series $0 < c < 1$ which relies on the coset construction of Goddard–Kent–Olive **[15]**. The discrete series at $c = 1$ can be regarded as a limiting case of this construction. The complexification $\mathfrak{g}$ of the Lie algebra of $SU(2)$ can be identified with the 2×2 complex matrices of trace zero. It is closed under taking adjoints and the Lie algebra of $SU(2)$ can be identified with the skew–adjoint matrices of trace zero. The affine Lie algebra $L\mathfrak{g}$ is defined as the space of trigonometric polynomial maps into $\mathfrak{g}$ with the pointwise bracket. It has generators $X_n = e^{in\theta}X$ ($X \in \mathfrak{g}$) with commutation relations $[X_m, Y_n] = [X, Y]_{m+n}$. The Witt algebra acts naturally by differentiation, so that $[\ell_n, X_k] = -kX_{k+n}$. In particular ℓ_0 defines a derivation $[d, X_n] = -nX_n$. The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is defined as a central extension of $L\mathfrak{g} \rtimes \mathbb{C}d$ by $\mathbb{C}$. It has generators $X(n)$, D and C (central) satisfying the commutation relations

$$[X(m), Y(n)] = [X, Y](m + n) + m\delta_{n+m,0}\mathrm{Tr}(XY) \cdot C, \;\; [D, X(n)] = -nX(n).$$

The Lie subalgebra generated by the $X(n)$ and C is denoted $\mathcal{L}\mathfrak{g}$: it is a central extension of $L\mathfrak{g}$ by $\mathbb{C}$. Again we can define positive energy unitary representations on an inner product space $\mathcal{H}$ by requiring that $C^* = C$ acts as a scalar ℓI, $D^* = D$. $X(n)^* = X^*(-n)$ and that $\mathcal{H} = \oplus_{n\geq 0}\mathcal{H}(n)$, where the $\mathcal{H}(n)$'s are finite–dimensional eigenspaces of D corresponding to the eiegnvalue n. Any such representation decomposes as a direct sum of irreducible positive energy representations of the same form (possibly after adjusting D by subtracting a positive scalar). For infinite–dimensional irreducible representations, ℓ must be a positive integer, called the *level*, and $\mathcal{H}(0)$ an irreducible representation of the Lie subalgebra $\mathfrak{g}(0) = \mathfrak{g}$. Each such V_j is specified by a non–negative half–integer spin j with $\dim V_j = 2j + 1$. At level ℓ, the only spins that occur are those satisfying $0 \leq j \leq \ell/2$. Yhe Segal–Sugawara construction shows that in each irreducible positive energy representation operators L_n can be defined with

$$L_0 = D + h \cdot I, \quad h = \frac{j^2 + j}{2(\ell + 2)}$$

such that $[L_n, X(m)] = -mX(n + m)$, $L_n^* = L_{-n}$ and

$$[L_m, L_n] = (m - n)L_{n+m} + \frac{m^3 - m}{12}c_\ell \cdot I$$

where $c_\ell = 3\ell/(\ell + 2)$. The character of the representation is defined as the formal sum

$$\chi_{\ell,j}(q, g) = q^h \sum_{n\geq 0} q^n \mathrm{Tr}_{\mathcal{H}(n)}(g),$$

where $g \in SU(2)$. Since this only depends on the conjugacy class of g, as usual we may take

$$g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},$$

with $\zeta| = 1$. Note that formally

$$\chi_{\ell,j}(q, g) = \mathrm{Tr}_{\mathcal{H}}(gq^{L_0}).$$

These characters are given by the Weyl–Kac character formula (see **[23], [31]**)). In **[36]** this formula was proved purely in terms of unitary representations together using the supersymmetric operators of Kazama–Suzuki; see also **[26]**.

Let $\mathcal{H}_{\ell,j}$ denote the irreducible representation of level ℓ and spin j. The algebraic tensor product $\mathcal{H} = \mathcal{H}_{\ell,r} \otimes \mathcal{H}_{m,s}$ can be expanded as a direct sum

$$\mathcal{H} = \mathcal{H}_{\ell,r} \otimes \mathcal{H}_{m,s} = \bigoplus \mathcal{H}_{m+\ell,t} \otimes M_t,$$

where $t-r-s$ is an integer and M_t is a multiplicity space that can be identified with $\mathrm{Hom}_{\mathcal{L}\mathfrak{g}}(\mathcal{H}_{m+\ell,t},\mathcal{H})$. Now if $T\in \mathrm{Hom}_{\mathcal{L}\mathfrak{g}}(\mathcal{H}_{m+\ell,t},\mathcal{H})$ so too is L_nT-TL_n where the action on the right is by the Segal–Sugawara construction on $H_{m+\ell,t}$ and the action on the left on $\mathcal{H}=\mathcal{H}_{\ell,r}\otimes\mathcal{H}_{m,s}$ is given the tensor product of the Segal–Sugawara constructions on $\mathcal{H}_{\ell,r}$ and $\mathcal{H}_{m,s}$. Hence the operator defined on the tensor product as $\pi_\ell(L_n)\otimes I+I\otimes\pi_m(L_n)-\pi_{\ell+m}(L_n)$ acts on the multiplicity space M_t. This is the Goddard–Kent–Olive construction. It gives a representation of the Virasoro algebra with central charge $c=c_\ell+c_m-c_{\ell+m}$. In particular when $m=1$, we have a decomposition

$$\mathcal{H}=\mathcal{H}_{\ell,r}\otimes\mathcal{H}_{1,s}=\oplus\mathcal{H}_{\ell+1,t}\otimes M_t,$$

with $s=0$ or $1/2$, $r-t\in s+\mathbb{Z}$ and

$$c=\frac{3\ell}{\ell+2}+1-\frac{3(\ell+1)\ell}{\ell+3}=1-\frac{6}{(\ell+2)(\ell+3)}.$$

The character of the multiplicity space is determined directly by the Weyl–Kac characters at level 1, ℓ and $\ell+1$ and this gives the lowest eigenvalue of L_0 in M_t, which turns out to be $h=h_{p,q}(m)$ with $p=2r+1$ and $q=2t+1$. The representation $L(c,h)$ therefore occurs as a summand of M_t. This shows that for this value of c, all the h appearing in the list of Friedan–Qiu–Shenker are indeed unitary. To complete the proof we just have to show that M_t is irreducible.

This is accomplished by invoking the easy part of the Friedan–Qiu–Shenker theorem. Indeed as observed in **[36]**, their proof of the classification theorem splits naturally into two parts. The first part shows that for c fixed at $1-6/m(m+1)$, the values of h have to be one of those in their list. The proof requires the Kac determinant formula for $M=M(c,h)$ (see **[22]**). This computes the determinant of (v_i,v_j) where the v_i's run across the canonical monomial basis of $M(n)$. If c is fixed, this is a constant times a polynomial in h. On the other hand, since χ_{M_t} gives an upper bound for $\chi_{L(c,h)}$, it follows that $M=M(c,h)$ has a singular vector in $M(pq)$. The Kac determinant for $M(n)$ therefore must vanish when $n\geq pq$. This implies that $h-h_{p,q}(c)$ divides the Kac determinant if m is given by $c=1-6/m(m+1)$. By the structure of the Verma modules, it occurs with multiplicity at least $\mathcal{P}(n-pq)$ where $\mathcal{P}(n)$ is the partition function. Thus the Kac determinant is divisible by

$$\prod_{p,q;1\leq pq\leq n}(h-h_{p,q}(c))^{\mathcal{P}(n-pq)}.$$

Since it can be checked that this product has the same degree as the Kac determinant and they agree up to a constant in the highest power, the product above is proportional to the Kac determinant. We will use the same strategy of proof to establish a similar product formula, due to Feigin and Fuchs, for a polynomial arising elsewhere in the representation theory of the Virasoro algebra.

Finally to prove that M_t is irreducible, on the one had hand the Kac determinant formula gives a lower bound for the character of $\mathcal{H}=L(c,h)$. It follows that the $M_t(n)$ agrees with $\mathcal{H}(n)$ for $n<m(m+1)-(m+1)p+mq=M$. On the other hand if M_t is reducible it is a sum of representation $L(c,h')$ from the list of Friedan–Qiu–Shenker. But it is elementary to verify that $h'<M+h$ for all such h'. It follows that M_t must be irreducible.

The proof of the character formula for $c=1$ and $h=j^2$ proceeds similarly by indentifying a multiplicity space on which the Virasoro algebra acts and proving that it is irreducible. Graeme Segal **[32]** gave a direct proof of the first case when $h=0$. Our proof is different in this case. It avoids the Segal–Sugawara construction and the Kac determinant formula, relying instead on various well known aspects of the fermion–boson correspondence as tools, including elementary parts of the theory of vertex algebras (see **[6]**, **[12]**, **[14]**, **[24]**).

Indeed in this case a single complex fermion field is given by a set of operators (e_n), d acting on an inner product space $\mathcal{H}$ subject to the anticommutation relations

$$e_me_n+e_ne_m=0,\ e_me_n^*+e_n^*e_m=\delta_{m+n,0}\cdot I,\ d=d^*,\ [d,e_n]=-(n+\frac{1}{2})e_n.$$

Slightly relaxing the positive energy condition, we require that $\mathcal{H}$ has a decomposition $\mathcal{H}=\oplus\mathcal{H}(r)$ where r runs over non–negative half integers with $\mathcal{H}(r)=\{\xi:d\xi=r\xi\}$ finite–dimensional. There is essentially only

one such irreducible representation on an inner product space $\mathcal{F}_f$ generated by a vacuum vector $\Omega \in \mathcal{F}_f(0)$. $\mathcal{F}_f$ is called *fermionic Fock space* and can be identified with an exterior algebra.

A single boson field is formed by operators (a_n), d acting on an inner product space $\mathcal{K}$ and satisfying the commutation relations

$$a_m a_n - a_n a_m = m\delta m + n, 0 \cdot I,\ a_n^* = a_{-n},\ d = d^*,\ [d, a_n] = -na_n.$$

Again there is an essentially unique irreducible positive energy representation, this time with non–negative integer eigenvalues for d. It is generated by $\Omega \in \mathcal{K}(0)$ satisfying $a_0\Omega = \mu\Omega$.A unitary charge operator U can be added to this bosonic system satisfying the additional relations

$$Ua_nU^* = a_n + \delta_{n,0}I, \quad UdU^* = d + a_0 + \frac{1}{2}I.$$

The "discrete" system U, a_0 and d has an essentially unique positive energy representation with d having non–negative half integer eigenvalues. From this it follows that the charged boson system (a_n, U, d) has an essentially unique positive energy representation on an inner product space $\mathcal{F}_b$, obtained as a tensor product. $\mathcal{F}_b$ is called *bosonic Fock space* and can be identified with a symmetric algebra tensored with the algebraic group algebra of $\mathbb{Z}$.

Boson–fermion duality is the statement that there is a natural unitary isomorphism between $\mathcal{F}_f$ and $\mathcal{F}_b$ compatible with the operators, carrying Ω_f onto Ω_b. Identifying the spaces, the compatibility conditions state that d should be preserved and that

$$[a_n, e_m] = e_{n+m},\ Ue_nU^* = e_{n+1}.$$

On the other hand it is straightforward to see that there are essentially unique operators on $\mathcal{F}_f$ satisfying these relations and that they act irreducibly. In fact U arises as an explicit shift operator on $\mathcal{F}_f$ and the a_n's can be written as linear combination of operators of the form $e_ie_j^*$. These operators act irreducibly on $\mathcal{F}_f$, so in this sense charged bosons can be constructed from fermions. To proceed in the other direction, i.e. to construct fermions from charged bosons, requires the introduction of vertex operators. For $m \in \mathbb{Z}$, these are defined as formal power series in z and z^{-1} by the formula

$$\Phi_m(z) = U^m z^{-ma_0} \exp(\sum_{n<0} \frac{mz^n a_n}{n}) \exp(\sum_{n>0} \frac{mz^n a_n}{n}).$$

The fundamental identities are then

$$\Phi_1(z) = \sum e_n z^{-n-1},\ \Phi_{-1}(z) = \sum e_{-n}^* z^{-n}.$$

Identifying $\mathcal{F}_f$ and $\mathcal{F}_b$, which we write simply as $\mathcal{F}$, there are Virasoro operators L_n with $c = 1$ satisfying the covariance relations

$$[L_n, e_m] = -(m + \frac{1}{2}(m+1))e_{m+n},\ [L_n, a_m] = -ma_{m+n}. \tag{3}$$

The operators L_n can be written either as linear combination of operators of the form $e_ie_j^*$ or of the form a_ia_j. The identity (3) is a special case of the more general Fubini–Veneziano relation

$$[L_n, \Phi_m(z)] = z^{n+1}\Phi_m'(z) + \frac{m^2}{2}(n+1)z^n\Phi_m(z). \tag{4}$$

Note that the operators L_n commute with a_0 and so leave invariant its eigenspaces. The action on the non–zero eigenspaces turns out to be irreducible. The action on the zero eigenspace, however, is not irreducible. We will see that it is a direct sum of representations $L(1, m^2)$ with m a non–negative integer. Similarly the Virasoro algebra acts on the irreducible representation of (a_n), d with $a_0 = \frac{1}{4}I$, which it will turn out decomposes as a direct sum of representations $L(1, (m + \frac{1}{2})^2)$.

A natural way to understand and study these decompositions is by passing to $\mathcal{F}^{\otimes 2} = \mathcal{F} \otimes \mathcal{F}$. The description of $\mathcal{F}$ as an exterior algebra shows that there is a natural action on $\mathcal{F}^{\otimes 2}$ of the group $SU(2)$ and thus its Lie algebra. This action commutes with the operator $d \otimes I + I \otimes d$. Conjugating the action of $a_n \otimes I$ by elements of $SU(2)$ then leads to operators $E_{ij}(n)$ with $E_{11}(n) = a_n$ satisfying $E_{ij}(n)^* = E_{ji}(-n)$ and

$$[X(m), Y(n)] = [X, Y](n+m) + m\delta_{m+n,0} \mathrm{Tr}\, XY,$$

if $X = \sum x_{ij} E_{ij}$ and $Y = \sum y_{ij} E_{ij}$. In particular taking $X \in \mathfrak{sl}_2$, the matrices with zero trace, we get a positive energy representation of the Kac–Moody Lie algebra at level 1. These operators commute with the operators $A_n = \frac{1}{2}(a_n \otimes I + I \otimes a_n)$. In particular the A_n's commute with the operators $B_m = H(m)$, $E(m)$ and $F(m)$ where $H = \frac{1}{2}(E_{11} - E_{22})$, $E = E_{12}$ and $F = E_{21}$ is the standard basis of $\mathfrak{sl}_2$. These operators are related to the vertex operators by the formulas

$$\Phi_1(z) \otimes \Phi_{-1}(z) = \sum E(n) z^{-n-1},\ \Phi_{-1}(z) \otimes \Phi_1(z) = \sum F(n) z^{-n-1}, \tag{5}$$

identities originally due to Frenkel–Kac **[10]** and Segal **[32]**.

The space $\mathcal{F}^{\otimes 2}$ decomposes as a sum of two components

$$\mathcal{F}^{\otimes 2} = \mathcal{H}_0 \otimes M_0 \oplus \mathcal{H}_{1/2} \otimes M_{1/2},$$

with the affine Kac–Moody algebra acting irreducibly on $\mathcal{H}_j$ and the bosonic operator irreducibly on the multiplicity spaces M_j. The action of the Virasoro algebra on $\mathcal{F}^{\otimes 2}$ preserves this decomposition. Its actions on the tensor factors coincides with the natural action associated with (B_n) and (A_n). The action on H_j commutes with the action of $SU(2)$ and satisifies $[L_n, X(m)] = -mX(n+m)$. Each space $\mathcal{H}_j$ may be further decomposed according to the action of $SU(2)$

$$\mathcal{H}_j = \bigoplus_{i \geq 0,\, i-j \in \mathbb{Z}} V_i \otimes \mathcal{K}_{ij}.$$

The operators L_n commute with $SU(2)$, so preserve the multiplicity spaces $\mathcal{K}_{ij}$, each of which contains a copy of $L(1, i^2)$ generated by a singular vector in the lowest energy space. This action on the multiplicity spaces can be regarded as a limiting case of the coset construction of Goddard–Kent–Olive. The singular vectors were described explicitly by Goldstone **[21]** and we shall refer to them as *Goldstone vectors*. Goldstone's formulas were shown by Segal **[32]** to be a direct consequence of the vertex operator formulas in (5). Although the formulas can be developed without any advanced theory of symmetric functions **[27], [29]**, the simplest way to describe them involves the combinatorics of the Weyl character formula for $U(n)$ **[40]**. In fact setting $X_n = a_{-n}$ and regarding these as the symmetric functions with signature $(n, 0, 0, 0, \ldots)$, the Goldstone vectors correspond to the symmetric functions X_f with signature $(k+m, k+m, \ldots, k+m, 0, 0, \ldots)$, where $k+m$ appears k times applied to a singular vector for the a_n with $n > 0$. (If we took $X_k' = \sum_i z_i^k$, then X_f' would just be $\det z_j^{f_i+n-i} / \det z_j^{n-i}$, the character of the irreducible representation V_f of $U(n)$ with signature f.) Goldstone conjectured that these vectors were the only singular vectors in $\mathcal{H}_j$ so that consequently the Virasoro algebra acted irreducibly on the multiplicity spaces. Since $\mathrm{ch}\, \mathcal{K}_{ij} = (q^{i^2} - q^{(i+1)^2}) \cdot \varphi(q)$, this establishes the character formulas for $L(1, i^2)$.

To prove this directly, it suffices to show that in $\mathcal{H}_j$ the Goldstone vectors are the only singular vectors in a fixed eigenspace $\mathcal{K}$ of $H(0)$. It is easy to check that $\mathcal{K}$ is an irreducible representation of the $H(n)$'s and that, up to scalar multiples, there is at most one singular vector at any fixed energy level in $\mathcal{K}$. Moreover as module over $\mathfrak{vir}$, $\mathcal{K}$ is a direct sum of irreducible representations, with one component for each singular vector. If there were a component $\mathcal{K}'$ isomorphic to $L(1, p)$ with p not of the form $(j+k)^2$, then by irreducibility, for some component $\mathcal{K}''$ generated by a Goldstone vector, $H(a)\mathcal{K}''$ would have to have a non–zero projection on $\mathcal{K}'$. But if P' and P'' denotes the orthogonal projections onto $\mathcal{K}'$ and $\mathcal{K}''$, then

$$\Psi(z) = P' H(z) P''$$

would give a formal power series of operators from $\mathcal{K}''$ to $\mathcal{K}'$ satisfying

$$[L_k, \Psi(z)] = z^{k+1} d\Psi'(z). \tag{6}$$

Such family of operators is a special case of what is called a *primary field*. Writing $\Psi(z) = \sum \psi(n) z^{-n}$, (6) can be rewritten

$$[L_k, \psi(n)] = -n\psi(n+k). \tag{7}$$

This prompts the introduction of the *density modules* $V_{\lambda,\mu}$ giving the natural representations of the Witt algebra on expressions of the form $f(\theta)e^{i\mu\theta}(d\theta)^\lambda$, with f a trigonometric polynomial. Multiplication and integration over the circle gives a natural pairing with $V_{1-\lambda,-\mu}$. A primary field, for appropriate λ and μ, then defined to be a linear map $\mathcal{K}'' \otimes V_{\lambda,\mu} \to \mathcal{K}'$ commuting with the action of the Virasoro algebra. By duality it is the same as an equivariant map $\mathcal{K}'' \to \mathcal{K}' \otimes V_{\lambda',\mu'}$, where $\lambda' = 1-\lambda$ and $\mu' = -\mu$. The formula (6) is then replaced by the more general relation

$$[L_k, \Psi(z)] = z^{k+1} d\Psi'(z) + \lambda z^k \Psi(z). \tag{8}$$

Returning to (6) and (7), the action there is simply on functions, so that $\lambda = 0 = \mu$. But then $V_{0,0}' = V_{1,0}$, the space of differentials $f(\theta)\, d\theta$. On the other hand if $\mathcal{K}''$ is isomorphic to $L(1,i^2)$ then it is the quotient of a Verma module $M(1,i^2)$ by a submodule containing at least one singular vector w of energy $(i+1)^2$. If v is the lowest energy vector of $M(1,i^2)$, then w has the form Pv for some non–commuting polynomial in L_{-k} ($k \geq 1$). Taking the component of the lowest energy vector of K_1, it is then easy to see that a necessary condition for the existence of a primary field $\Psi(z)$ is that the action of P on one of the standard basis vectors of $V_{\lambda',\mu'}$ must be zero. But the action of the Witt algebra is given in this basis as shift operators weighted by polynomials, so the condition is equivalent to the vanishing of a polynomial. A formula for this polynomial was given without proof by Feigin–Fuchs [**8**]. In this particular case where $c = 1$, sufficiently many factors of the polynomial can be produced by explicitly exhibiting enough primary fields to determine completely the polynomial. (These are constructed as operators between different $L(1,i^2)$ by compressing $\Phi_a(z) \otimes \Phi_b(z)$.) Irreducibility then follows by noting that this polynomial does not vanish for the hypothetical field $\Psi(z)$ constructed above.

The proof that there are sufficiently many primary fields is essentially the first step in showing that under fusion the representations $L(1,i^2)$ for i a non–negative half integer behave exactly like the irreducible representations V_i of $SU(2)$ under tensor product (see [**38**]). Now if $\Psi(z)$ is a primary field and $L_{-1}w = 0$, we see that $F(z) = \Psi(z)w$ satisfies the so–called "equation of motion"

$$\frac{dF}{dz} = L_{-1}F,$$

so that

$$F(z) = e^{zL_{-1}}\Psi(0)\Omega.$$

This identity allows the existence of sufficiently many primary fields to be reduced to checking that $L_1^{|f|} X_f v \neq 0$ where v is a singular vector of the $H(n)$'s for $n > 0$ with $H(0)v = pv$ with p a poitive integer. But it is easy to see that $(|f|!)^{-1} L_1^{|f|} X_f v = av$ for some constant a. Like X_f, the value of a is given by a determinant, but this time of binomial coefficients. It can be calculated as an explicit product using either the Weyl dimension formula or an elementary matrix computation. In particular, in the case of interest, it does not vanish: indeed remarkably a equals the dimension of the irreducible representation V_f of $U(p)$ when this makes sense and vanishes otherwise. This can be summarised by saying that, if the Goldstone vectors are given formally by formulas relating to the Weyl character formula, then their non–degeneracy properties follow from the corresponding Weyl dimension formula.

Sections 2 to 8 give a self–contained step–by–step account of this method of establishing the character formula for $L(1,j^2)$. Appendix A contains a alternative direct verification of the Fubini–Veneziano relations for vertex operators. In Appendices B and C, we give an alternative approach to the product formula of Feigin–Fuchs using an elegant algorithm of Bauer, Di Francesco, Itzykson and Zuber ([**3**], [**7**]) for determining the polynomial P, related to an earlier formula of Benoit and St Aubin [**4**]: we give a simplification of their treatment in Appendix A. The most general Feigin–Fuchs formula can be proved using the coset construction for primary fields for the unitary discrete series with $0 < c < 1$. Indeed the primary fields for the affine Kac–Moody algebra $\mathfrak{sl}_2$ can be constructed explicitly and then used to contruct primary fields for the Virasoro algebra [**28**], [**39**]. In this case the irreducible representations for $c = 1 - 6/m(m+1)$ with m large are

indexed by non–negative half integers (i,j) with $i-j$ and integer. The corresponding representation has $h=h_{p,q}(m)$ with $p=2i+1$ and $q=2j+1$. The singular vector at level pq acts on the density module as a multiple of $\prod(h-h_{p',q'}(m))$ where $V_i\otimes V_{i_1}=\bigoplus V_{i'}$ and $V_j\otimes V_{j_1}=\bigoplus V_{j'}$. Other methods of proving the Feigin–Fuchs formula are given in **[7]** and **[13]**.

In Section 9 we desribe a simple method for determining the character formula for $L(1,j^2)$ using only the Kac determinant formula and the Jantzen filtration **[20]**. It is generalised in Section 10 to the case $0<c<1$ and gives a different method of proof to that of Astashkevich **[1]**. Let $A(x)=\sum_{i\geq 0}A_ix^i$ be an analytic family of non–negative self–adjoint matrices defined for x real and small on $V=\mathbb{R}^n$ or $\mathbb{C}^n$. We define a filtration $V^{(i)}$ $(i\geq 0)$ by $V^{(0)}=V$ and for $m\geq 1$

$$V^{(m)}=\bigcap_{i=0}^{m-1}\ker A_i.$$

In the examples $A(x)$ will be invertible for $x\neq 0$ in a neighbourhood of 0. It follows that $V^{(m)}=(0)$ for m sufficiently large. The fundamental identity, which is easy to prove, is that the order of 0 as a root of $\det A(x)$ equals

$$\sum_{i\geq 1}\dim V^{(i)}.$$

Taking the standard inner product on V, we can define a Hermitian form on V by $(v_1,v_2)_x=(A(x)v_1,v_2)$. This filtration has an important functorial property. Suppose we are given two such space $(V,A(x))$ and $(W,B(x))$ and in addition analytic maps $X(x):V\to W$, $Y(x):W\to V$ such that

$$(X(x)v,w)_x=(v,Y(x)w)_x.$$

It then follows easily that $X(0)$ and $Y(0)$ induce maps between $V^{(i)}$ and $W^{(i)}$ preserving the canonical Hermitian forms induced by A_i and B_i.

Now let $M=M(1+x,j^2)$. Then $V=M(n)$ is independent of x with a canonical basis, so can be identified with $\mathbb{C}^{\mathcal{P}(n)}$. We consider the Jantzen filtration for $M=M(1,j^2)$ associated with the parameter x in $c=1+x$. The last property above implies that each $M^{(i)}\equiv\oplus M(n)^{(i)}$ is a submodule for the Virasoro algebra and comes with an invariant Hermitian form. By the Kac determinant formula we can explicitly compute $\sum_{i\geq 1}\operatorname{ch}M^{(i)}$. Now the determinant formula or the coset construction above show that $M(1,j^2)$ has a filtration by Verma modules $M(1,(j+k)^2)$ for $k\geq 1$. Let $M^{(n_k)}$ be the last term in the Jantzen filtration containing $M(1,(j+k)^2)$; it turns out that (n_k) is strictly increasing (we set $n_0=0$). But a lower bound for the character sum above is given by $\sum_{k\geq 1}(n_k-n_{k-1})\mathrm{ch}\,M(1,(j+k)^2$ and hence $\sum_{k\geq 1}\operatorname{ch}M(1,(j+k)^2$. However this sum actually equals the character sum. Hence $M^{(k)}=M(1,(j+k)^2)$ and hence $L(1,j^2)=\varphi(q)\cdot(q^{j^2}-q^{(j+1)^2})$.

The proof for the discrete series is similar but slightly but more intricate. Let $c=1-6/m(m+1)$ and $M=M(c(m),h_{p,q}(m))$ with $1\leq q\leq p\leq m-1$. Again the Kac determinant formula produces a sequence of singular vectors in M which generate a series of Verma submodules A_i,B_i for $i\geq 1$ such that A_i and B_j both contain A_k and B_k for $k>i$. Thus $A_k\cap B_k\supseteq A_{k+1}+B_{k+1}$. Again, this time using $x=h-h_{p,q}(m)$ as a parameter, we can compute the character sum $\sum_{i\geq 1}\operatorname{ch}M^{(i)}$ as well as that for A_1 corresponding to the singular vector with energy $h+pq$. Analogous inequalities to those above prove that $A_k\cap B_k=A_{k+1}+B_{k+1}$ for $k\geq 1$ and that $M^{(k)}=A_k+B_k$ for $k\geq 1$. Using the isomorphisms $A_k\oplus B_k/(A_{k+1}+B_{k+1})=A_k+B_k$, the character formula for $L(c(m),h_{p,q}(m))=M/(A_1+B_1)$ follows.

Although we have not checked this, it seems likely that variants of the two proofs above can be used to simplify substantially the proof of Feigin and Fuchs. Such a simplification had been given by Astashkevich **[1]** for all cases except $\mathrm{III}_{\pm}^{00}$. The method of Feigin–Fuchs in this case was considerably more complicated than what is required for the discrete series $c<1$ (the case III_-) and invokes the Riemann–Roch theorem for holomorphic vector bundles on the sphere.

We end this overview by briefly describing the Feigin and Fuchs' original proof of the character formula for $L(1,j^2)$ in **[8]**—their case III_{-}^{00}. In addition to a knowledge of the discrete series characters and their

Jantzen filtrations, their proof requires the fact that the formula for the singular vector is a polynomial in t of degree $2j+1$, with constant term L_{-1}^{2j+1} and leading coefficient $[(2j)!]^2 L_{-2j-1}$. This was proved indirectly by Astashkevich and Fuchs **[2]** for $M(1,j^2)$. In this case we show in Appendix A that it is an easy consequence of the formula of Bauer, Di Francesco, Itzyksohn and Zuber **[3]** for the singular vector of $M(h_{2j+1,1}(t), 13-6t-6t^{-1})$.

Instead of the Riemann–Roch theorem, all that is needed is the classical theory of holomorphic vector bundles on the Riemann sphere. Recall that according to the Grothendieck–Birkhoff theorem, any such bundle can be written uniquely as a sum of line bundles, each classified by their degree (the sum with multiplicitied of a the degrees of poles and zeros of any meromorphic section). Grothendieck's original proof **[19]** was algebraic–geometric using the language of divisors, although it was soon realized that the theorem followed from the Birkhoff factorization theorem (see **[16], [31], [38]**). Indeed it is known **[9]** that a rank n holomorphic vector bundle is trivial when restricted to either of the open discs $z < R$ and $|z| > r$ with $r < 1 < R$ (see Appendix D for a short analytic proof). But then it is specified by a holomorphic map f from the annulus $r < |z| < R$ into $GL_n(\mathbb{C})$. By Birkhoff's factorization theorem the restriction of f to the circle $|z| = 1$ can be written $f(z) = f_-(z)D(z)f_+(z)$ where $D(z)$ is a matrix with entries z^{n_i} on the diagonal, $f_\pm(z)$ is a function holomorphic on $D_\pm(z) = \{z : |z|^{\pm 1} < 1\}$ and smooth on its closure. The matrix $D(z)$ is uniquely determined up to the order of its entries. Since f extends to $r < |z| < R$ it follows immediately from the Schwarz reflection principle that $f_+(z)$ and $f_-(z)$ extend to homolorphic function on $|z| < R$ and $|z| > r$. But then the clutching function on the annulus can be replaced by $D(z)$ which corresponds to a direct sum of holomorphic line bundles. (Each of these can in turn be understood in terms of holomorphic sections of a homogeneous line bundle for the group $G = SU(2)$, identifying the Riemann sphere S with G/T, where T is the subgroup of diagonal matrices.)

Given a representation such as $L(1,j^2)$, the method of Feigin and Fuchs proceeds as follows. Let $r = 2j+1$. At each point of the curve C given by $\varphi_{r,1}(c,h) = 0$, there is a singular vector of energy $2j+1$, in the Verma module $M = M(c,h)$. The action of the operators L_{-n} for $n > 0$ is independent of c and h so the Verma module can be identified along the curve. The curve on the other hand can be parametrised by $c = 13 - 6t - 6t^{-1}$ and $h = (j^2+j)t - j$ with $t \in \mathbb{C}^*$. The singular vector $w(t) \in M(2j+1)$ depends polynomially on t (see Appendix B) and thus defines a line bundle on the Riemann sphere: its degree d is determined by the degree of the polynomial, since the constant term is non–vanishing. Let W_t be the Verma submodule of M generated by $w(t)$. At any fixed energy level k it is a sum of line bundles of degree d. So $M(k)/W_t(k)$ defines a holomorphic vector bundle E over the Riemann sphere, which is a sum of line bundles with the sum of the degrees determined. But if the rank of E is m, $F = \lambda^m E$ is a line bundle of known degree. Now the Kac determinant formula implies that generically M/W_t is irreducible on the curve $\varphi_{r,1}(c,h) = 0$. The Kac determinant in the quotient produces a meromorphic section of F^*, the line bundle dual to F. The degree of F^* is known. It can also be computed in terms of the order of poles and zeros of the meromorphic section. The contributions from 0 and ∞ can be computed directly, since there are two explicit bases independent of t in V complementary to W_t for $t \neq 0, \infty$. The computation reduces to that of a prinicipal minor where the diagonal terms dominate. Only real points given by the intersection of the curve with other curves $\varphi_{p,q}(c,h) = 0$ with $pq \leq N$ give other contributions. For $c \neq 1$, these correspond to Verma modules of type III_- having the same structure of singular vectors as the discrete series with $0 < c < 1$. These will have a singular vector of type $(r,1)$ amongst the other singular vectors (possibly of lower energy). Since the structure of these modules is known explicitly, the Jantzen filtration on M/W_t corresponding to the (real part of the) curve C can be computed and hence the contribution from these points. Adding up these contributions, it then follows that the contribution from $(1,j^2)$ is 0, as required.

2. Constructions of positive energy representations. We define the bosonic algebra $\mathfrak{a}$ to be the Lie algebra with basis a_n $(n \in \mathbb{Z})$, c and d, with non–zero brackets

$$[a_m, a_n] = m\delta_{m+n,0} \cdot c, \quad [d, a_n] = -na_n.$$

(Thus c is central.) A unitary positive energy representation of $\mathfrak{a}$ consists of an inner product space $\mathcal{H}$ which can be written as the orthogonal algebraic direct sum of finite–dimensional spaces $\mathcal{H}(n)$ $(n \geq 0)$

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}(n)$$

on which $\mathfrak{a}$ acts such that $d\xi = (n+h)\xi$ for $\xi \in \mathcal{H}(n)$, with $h \in \mathbb{R}$, $c = I$ and $a_n^* = a_{-n}$. d is called the *energy operator* and an element of $\mathcal{H}(n)$ is said to have energy $h+n$. Note that replacing d by $d+tI$ gives another essentially equivalent representation.

Note that if an operator a satisfies the canonical commutation relations $a^*a - aa^* = \mu I$ and $a^*v = 0$, then an easy induction argument shows that

$$(a^m v, a^n v) = \delta_{m,n} m! \mu^m. \tag{1}$$

Now let $V = \mathbb{C}[x_1, x_2, \ldots]$ and for $n > 0$ define operators $a_{-n}p = x_n p$, $a_n p = n\partial_{x_n} p$ and $a_0 p = \lambda p$. We define $\deg x_n = n$ and let D be the operator which multiplies a monomial by its degree, so that

$$D = \sum_{n \geq 1} n \cdot x_n \partial_{x_n}. \tag{2}$$

This gives a representation of the Lie algebra $\mathfrak{a}$. Since $p(x_1, x_2, \ldots) = p(a_1, a_2, \ldots)1$, repeated applications of (1) show that it is unitary for the inner product making monomials orthogonal with

$$\|x_1^{m_1} x_2^{m_2} \ldots\|^2 = \prod m_n!\, n^{m_n}.$$

It is irreducible because any non–trivial $\mathfrak{a}$–submodule must contain a non–zero vector p of lowest energy. Thus $a_n p = 0$ for all $n > 0$, i.e. $\partial_{x_n} p = 0$ for all $n > 0$. But then p is a constant and the submodule must be the whole of V. Conversely if W is any positive energy unitary representation, then by (1) any lowest energy vector unit w will satisfy $(p(a_{-1}, a_{-2}, \ldots)w, q(a_{-1}, a_{-2}, \ldots)w) = (p, q)$, the inner product on V constructed above. Let W_1 be the $\mathfrak{a}$–module generated by w. Adjusting a_0 and d by constants if necessary, it follows from (1) that W_1 is unitarily equivalent to V by the map $Up = p(a)w$. The orthogonal complement of W_1 is also a positive energy unitary representation so likewise contains an irreducible submodule W_2 of the same form. Continuing in this way, the positive energy condition shows that we can write $W = \bigoplus_{m \geq 1} W_m$, an orthogonal direct sum where each W_m is isomorphic to V but with a_0 acting as $\lambda_m I$ and d as $D + h_m I$. Thus λ_m and h_m are the eigenvalues of a_0 and d on the lowest energy vector w_m: clearly this determines the representation uniquely. Representations of the Virasoro algebra can be define on these spaces by the generalising the formula (1) as follows:

$$L_0 = \frac{1}{2}a_0^2 + \sum a_{-n}a_n, \;\; L_n = \frac{1}{2}\sum_{r+s=n} a_r a_s \, (n \neq 0).$$

Note that in the second formula the operators a_r and a_s commute. By definition L_n changes energy by $-n$. It is easy to check that $[L_n, a_m] = -m a_{m+n}$ and hence that $A_{m,n} = [L_m, L_n] - (m-n)L_{m+n}$ commutes with all the $a_j's$. It leaves each W_k invariant. By uniqueness of the lowest energy vector in W_k, it must act as a scalar $\nu_{m,n}$ and therefore preserve energy. Thus $A_{n,m} = 0$ unless $n = -m$. Otherwise for $n > 0$

$$\begin{aligned}(([L_n, L_{-n}] - 2nL_0)w_k, w_k) &= -n(a_0^2 w_k . w_k) + \frac{1}{2}(L_n \sum_{r+s=-n} a_r a_s w_k, w_k) \\ &= -n(a_0^2 w_k, w_k) + \frac{1}{2}\sum_{r+s=-n} -r(a_{n+r}a_s w_k, w_k) - s(a_r a_{n+s} w_k, w_k) \\ &= \frac{1}{2}\sum_{t=1}^{n} t(n-t) \\ &= (n^3 - n)/6.\end{aligned}$$

Thus

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{m^3 - m}{12}I.$$

We now make similar constructions using the canonical anticommutation relations. In this case the notion of positive energy representation has to be slightly generalised to allow the eigenvalues of the energy

operator to have the form $n+h$ with $n \in \frac{1}{2}\mathbb{Z}$. We define V to be the inner product space spanned by the mutually orthogonal unit vectors

$$e_{i_1} \wedge e_{i_2} \wedge e_{i_2} \wedge \cdots, \tag{3}$$

with $i_k \in \mathbb{Z}$, $i_1 < i_2 < i_3 < \cdots$ and $i_{k+1} = i_k + 1$ for k sufficient large. We define the vacuum vector by $\Omega = e_0 \wedge e_1 \wedge e_2 \wedge \cdots$ and make the e_i's act by exterior mutiplication. It is easily checked that the e_j^*'s act by interior multiplication and that

$$e_m e_n^* + e_n^* e_m = \delta_{m,n} I, \;\; e_m e_n + e_n e_m = 0.$$

The space V can be canonically identified with the exterior algebra in the anticommuting variables e_{-n} for $n > 0$ and e_n^* for $n \geq 0$. We can define the energy and charge of a basis vector in (3) as follows. Firstly it can be regarded as a monomial in the exterior algebra in the e_{-n}'s ($n > 0$) and the e_n^* ($n \geq 0$). A term e_{-n} or e_n^* contributes $n - 1/2$ to the energy and $+1$ and -1 to the charge. The vector

$$\Omega = \Omega_0 = e_0 \wedge e_1 \wedge e_2 \wedge \cdots$$

has energy 0 and charge 0. Let d and a be the operators multiplying the basis vectors by their energy and charge respectively. Thus $a^* = a$ and $d^* = d$. Morevoer $[d, e_i] = -(i + 1/2)e_i$ and $[a, e_i] = e_i$. Taking adjoints we get $[d, e_i^*] = (i + 1/2)e_i^*$ and $[a, e_i^*] = -e_i^*$. Alternatively a basis vector can be written

$$v = e_{m_1} \wedge \cdots \wedge e_{m_j} \wedge e_{j+k} \wedge e_{j+k+1} \wedge \cdots,$$

where $m_1 < m_2 < \cdots < m_j < k + j$. Thus $m_i < k + i$. The vector will have charge $-k$ and energy $\sum_{i=1}^{j}(k + 1/2 - i - m_i)$. Indeed the vector

$$\Omega_k = e_k \wedge e_{k+1} \wedge \cdots$$

has charge $-k$ and energy $k^2/2$. The operators $e_a e_b^*$ commute with a so preserve charge. Since v is obtained from Ω_k by successively applying the operators $e_{m_i} e_{k-i}^*$, which raises the energy by $k - i - m_i$, it follows that v has energy $\sum_{i=1}^{j}(k - i - m_i)$. We define

$$a_k = \sum_{n-m=k} e_n e_m^*, \;\; a_0 = \sum_{n \geq 0} e_{-n} e_{-n}^* - \sum_{n > 0} e_n^* e_n$$

and

$$L_k' = \sum_{n-m=k} -(m + 1/2 + k/2) e_n e_m^*, \;\; L_0' = \sum_{n>0} (n - \frac{1}{2}) e_{-n} e_{-n}^* + \sum_{n \geq 0} (n + \frac{1}{2}) e_n^* e_n.$$

It is straightforward to check that $a_n^* = a_{-n}$, $(L_n')^* = L_{-n}'$ and

$$[L_n', e_m] = -(m + \frac{n}{2} + \frac{1}{2}) e_{n+m}, \;\; [L_n', e_m^*] = (m - \frac{n}{2} - \frac{1}{2}) e_{m-n}^*, \;\; [a_n, e_m] = e_{n+m}, \;\; [a_n, e_m^*] = -e_{m-n}^*.$$

The operators $d - L_0'$ and $a - a_0$ commute with the e_n's and e_n^*'s and annihilate Ω. By cyclicity of Ω, it follows that $d = L_0'$ and $a = a_0$. The operator $[a_n, a_m]$ commute with the e_k's and e_k^*'s and raise energy by $-n - m$. If $n + m < 0$ it must annihilate Ω and hence vanish. Since $[a_n, a_m]^* = [a_{-m}, a_{-n}]$. it follows that $[a_n, a_m] = 0$ if $n + m \neq 0$. Now up to a scalar multiple Ω is the unique vector of energy 0, so that if $n > 0$, $[a_n, a_{-n}]\Omega = \lambda_n \Omega$; by cyclicity $[a_m, a_n] = \lambda_n \delta_{m+n,0} I$. To evaluate λ_n we have

$$\lambda_n = ([a_n, a_{-n}\Omega, \Omega) = -\|a_{-n}\Omega\|^2 = n.$$

Thus

$$[a_m, a_n] = m\delta_{n+m,0} I.$$

Similarly $[L'_m, L'_n] - (m-n)L'_{m+n}$ commutes with the e_k's and e_k^*'s and raise energy by $-n-m$. Using adjoints as before, it must be 0 if $n+m \neq 0$ and a scalar μ_m if $m+n=0$. Clearly $\lambda_{-m} = -\lambda_m$ using adjoint and if $n > 0$

$$\begin{aligned}\lambda_n &= ([L'_n, L'_{-n}]\Omega, \Omega) \\ &= \|L'_{-n}\Omega\|^2 \\ &= \sum_{m=1}^{n} (m - n/2 - 1/2)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)^2}{4} - \frac{(n+1)^2 n}{2} \\ &= \frac{1}{12} n(n+1)[4n+2+3(n+1)-6(n+1)] = \frac{1}{12}(n^3 - n).\end{aligned}$$

Hence we have

$$[L'_m, L'_n] = (m-n)L'_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12} I.$$

Lemma (Gomes). *The only irreducible positive energy unitary representation of the Virasoro algebra with $c = 0$ is the trivial representation.*

Proof. Let Ω be a lowest energy unit vector with $L_0\Omega = h\Omega$ and $L_n\Omega = 0$ for $n > 0$. Let $\xi_1 = L_{-2n}\Omega$ and $\xi_2 = L_{-n}^2\Omega)$. Then $\|\xi_1\|^2 = 4nh$, $\|\xi_2\|^2 = 4n^2h(2h+n)$ and $(\xi_1, \xi_2) = 6n^2h$. Hence

$$\det(\xi_i, \xi_j) = 16n^3h^2(2h+n) - 36n^4h^2 = 4n^3h^2(8h-5n).$$

To be non–negative for all n we must have $h = 0$. But then unitarity implies that $L_{-n}\Omega = 0$ for all $n > 0$.

We have constructed the positive energy representation of the fermionic operators e_n and e_n^* on $\mathcal{F}$ with vacuum vectors Ω. We have constructed operators, bilinear in fermions, a_n and L'_n satisfying $[a_m, a_n] = m\delta_{m+n,0}I$, $[L'_n, e_m] = -me_{n+m}$ and $[L'_n, a_m] = -ma_{m+n}$. The L'_n's define a representation of the Virasoro algebra with $c = 1$. On the other hand we may also define operators L_n, bilinear in the bosons a_i, which give another representation of the Virasoro algebra with $c = 1$ satisfying $[L_n, a_m] = -ma_{m+n}$. The coset operators $L''_n = L'_n - L_n$ define a positive energy representation of the Virasoro algebra with $c = 0$ on the multiplicity spaces $M_i = \{\xi | a_0\xi = i\xi,\ a_n\xi = 0\ (n > 0)\}$. So $L_n = L'_n$ on M_i and hence everywhere.

Corollary. $L'_n = L_n$.

We now generalize this contruction to produce representations of the Virasoro algebra with $c = 1$ and $h = j^2$ with j a non–negative half–integer. The space $\mathcal{F}$ is canonically $\mathbb{Z}_2$–graded and the grading is compatible with its identification with an exterior algebra. The operators e_i and e_i^* act as odd operators. On the $\mathbb{Z}_2$–graded tensor product $\mathcal{F}_2 = \mathcal{F} \otimes \mathcal{F}$ there are operators $e_i^{(1)} = e_i \otimes I$ and $e_i^{(2)} = I \otimes e_i$. The space $\mathcal{F}_2$ can be identified with the exterior algebra on the anticommuting variables $e_{-n}^{(j)}$ $(n > 0)$ and $(e_n^{(j)})^*$ $(n \geq 0)$. If for $v = (\alpha, \beta) \in \mathbb{C}^2$ we set $v_i = \alpha e_n^{(1)} + \beta e_n^{(2)}$ and $v_i^* = \overline{\alpha}(e_n^{(1)})^* + \overline{\beta}(e_n^{(2)})^*$, then the natural actions of $G = U(2)$ on V and V^* (the conjugate space) extend to a unitary action π on the exterior algebra and hence $\mathcal{F}_2$. Evidently

$$v_n w_m + w_m v_n = 0,\ v_n w_m^* + w_m^* v_n = \delta_{n,m}(v,w)I,\ \pi(g)v_i\pi(g)^* = (gv)_i.$$

For $i, j = 1, 2$ we define

$$E_{ij}(k) = \sum_{n-m=k} e_n^{(i)}(e_m^{(j)})^*,\ E_{ij}(0) = \sum_{n\geq 0} e_{-n}^{(i)}(e_{-n}^{(j)})^* - \sum_{n>0}(e_n^{(j)}) * e_n^{(i)}.$$

Identifying E_{ij} with the canonical basis of $\operatorname{End} V = M_2(\mathbb{C})$, we can extend this definition by linearity to define operators $X(n)$ for $X \in \operatorname{End} V$. Note that these operators are even. We define $D = L_0 \otimes I + I \otimes L_0$, so that $[D, v_i] = -iv_i$ and $[D, X(n)] = -nX(n)$. More generally set $\mathcal{L}_n = L_n \otimes I + I \otimes L_n$. Thus

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \delta_{m+n,0}\frac{m^3-m}{6}I,\ [\mathcal{L}_m, v_n] = -nv_{n+m}.$$

Lemma. *(a)* $\pi(g)X(n)\pi(g)^* = (gXg^*)(n)$ *for* $g \in U(2)$.
(b) $E_{11}(n) = a_n \otimes I$, $E_{22}(n) = I \otimes a_n$.
(c) $X(n)^* = X^*(-n)$.
(d) $[X(n), v_i] = (Xv)_{i+n}$.
(e) $[\mathcal{L}_m, X(n)] = -nX(n+m)$.
(f) $[X(m), Y(n)] = [X,Y](m+n) + m\delta_{m+n,0} \cdot \mathrm{Tr}(XY)I$.

Proof. The first three identities are immediate. By complex linearity it suffices to check the remaining relations for X, Y skew–adjoint, i.e. in the Lie algebra of $U(2)$. Relation (d) is evident for diagonal operators using (b) and the relation $[a_m, e_n] = e_{n+m}$. Using (a) and the relation $\pi(g)v_i\pi(g)^* = (gv)_i$, it follows for skew–adjoint X, since we can choose $g \in U(2)$ so that gXg^* is diagonal. Similarly (e) is evident for diagonal X and follows for skew–adjoint X from the fact that the operators $\pi(g)$ and $\mathcal{L}_n$ commute, because $L_n = L_n'$. From (d), it follows that $[X(m), Y(n)] - [X,Y](m+n)$ commutes with the v_i's. Taking adjoints, it also commutes with the v_i^*'s. It changes energy by $-n-m$, so vanishes if $n+m \neq 0$. Otherwise, if $n = -m$, it preserves energy and therefore takes the vacuum vector $\Omega \otimes \Omega$ to a scalar multiple of itself. Since the vacuum vector is cyclic, it acts as this scalar everywhere, so that

$$[X(m), Y(n)] - [X,Y](m+n) = \delta_{m+n,0} \cdot \lambda_m(X,Y).$$

Evidently $\lambda_m(X,Y)$ is bilinear in X and Y and, conjugating the left hand side by $\pi(g)$, satisfies

$$\lambda_m(gXg^*, gYg^*) = \lambda_m(X,Y).$$

Thus $\lambda_m(X,Y) = \mathrm{Tr}((X \otimes Y)Z)$ where $Z \in \mathrm{End}\, V \otimes V$ commutes with $U(2)$, or equivalently, since Z automatically commutes scalars, with $SU(2)$. But as a representation of $SU(2)$, $V = V_{1/2}$ and $V_{1/2} \otimes V_{1/2} = V_0 \oplus V_1$. Thus by Schur's lemma, the space of invariant bilinear forms is two–dimensional, with an easily identified basis:

$$\lambda_m(X,Y) = \alpha_m \cdot \mathrm{Tr}(X) \cdot \mathrm{Tr}(Y) + \beta_m \cdot \mathrm{Tr}(XY).$$

Since $E_{11}(m) = a_m \otimes I$ and $E_{22}(n) = I \otimes a_n$ commute and $[a_m \otimes I, a_n \otimes I] = m\delta_{m+n,0}I$, it follows that $\alpha_m = 0$ and $\beta_m = m$, as claimed.

We have thus constructed an action of $\widehat{\mathfrak{u}_2}$ on $\mathcal{F}_2$ by operators $X(n)$. Let $L_n^{(1)} = L_n \otimes I$ and $L_n^{(2)} = I \otimes L_n$. From the above, if $H_i = E_{ii}$, then $L_n^{(i)}$ is the Virasoro algebra associated to the bosonic system $H_i(k)$ and $\mathcal{L}_n = L_n^{(1)} + L_n^{(2)}$ satisfies $[L_n, X(m)] = -mX(m+n)$. Set $H = \frac{1}{2}(H_1 - H_2)$, $K = \frac{1}{2}(H_1 + H_2)$, $E = E_{12}$ and $F = E_{21}$. Then the operators $K(n)$ commutes with the operators $E(m)$, $F(m)$ and $H(m)$. Let $L_n^{SU(2)}$ and $L_n^{U(1)}$ be the Virasoro operators constructed using the two oscillator algebras $(\sqrt{2}H(n))$ and $(\sqrt{2}K(n))$. Since $H_1^2 + H_2^2 = 2(H^2 + K^2)$, we have $L_n = L_n^{SU(2)} + L_n^{U(1)}$. Explicitly we have

$$L_0^{SU(2)} = H(0)^2 + 2\sum_{n>0} H(-n)H(n), \quad L_0^{U(1)} = K(0)^2 + 2\sum_{n>0} K(-n)K(n).$$

Since $K(n)$ commutes with $X(m)$ for $X \in \mathrm{Lie}\, SU(2)$, it follows that $[L_n^{SU(2)}, X(m)] = -mX(m+n)$. By construction, since the operators $H(m)$ leave any $\widehat{\mathfrak{sl}_2}$–submodule invariant, so do the Virasoro operators $L_n^{SU(2)}$; moreover $L_n^{U(1)}$ leaves invariant the multiplicity spaces of the distinct level 1 representations of $\widehat{\mathfrak{sl}_2}$.

There are at most two such representations, classified by their lowest energy spaces $\mathcal{H}(0)$, isomorphic to either V_0 or $V_{1/2}$. In fact let V be an irreducible $SU(2)$–submodule of $\mathcal{H}(0)$ and let $\mathcal{K}$ be the $\widehat{\mathfrak{sl}_2}$–submodule generated by V. By the commutation relations any monomial in the $X(n)$'s can be written as a sum of monomials of the form RDL with R a monomial in energy raising operators $X(-n)$ $(n > 0)$, D a monomial in constant energy operators $X(0)$, and L a monomial in energy lowering operators $X(n)$ $(n > 0)$. Hence $\mathcal{K}$ is spanned by products Rv. But then clearly $\mathcal{K}(0) = V$. By irreducibility $\mathcal{H} = \mathcal{K}$ and hence $\mathcal{H}(0) = V$. We claim that if $V = V_j$, then $j = 0$ or $1/2$. Indeed let $e = F(1)$, $f = E(-1)$ and $2h = [e,f] = (E_{22}(0) - E_{11}(0)) + I = -2H + I$. Thus $h^* = h$, $e^* = f$, $[h,e] = e$ and $[h,f] = -f$. Suppose that $\mathcal{H}(0) \cong V_j$ and that $v \in V_j$ satisfies $Hv = jv$. So $hv = (1-2j)v$ and $ev = 0$. By standard $\mathfrak{sl}_2$ theory,

it follows that $1-2j\geq 0$, i.e. $j=0$ or $j=1/2$. Note that if $\mathcal{H}(0)\cong V_j$ with $j=0$ or $j=1/2$, then each $\mathcal{H}(n)$ decomposes as a sum of integer ($j=0$) or half–integer spin ($j=1/2$). Since the decomposition of $\mathcal{F}_2$ contains both integer and half–integer spin representations of $SU(2)$, it follows that there are irreducible positive energy representations with $\mathcal{H}(0)=V_j$ ($j=0,1/2$).

To prove uniqueness, note that any monomial A in operators from $\widehat{\mathfrak{sl}_2}$ is a sum of monomials RDL with R a monomial in energy raising operators $X(-n)$ ($n>0$), D a monomial in constant energy operators $X(0)$, and L a monomial in energy lowering operators $X(n)$ ($n>0$). If $v,w\in\mathcal{H}(0)$, then the inner products (A_1v,A_2w) are uniquely determined by v,w and the monomials A_i. Indeed $A_2^*A_1$ is a sum of terms RDL and $(RDLv,w)=(DLv,R^*w)$, with R^* an energy lowering operator. Hence if $\mathcal{H}'$ is another irreducible positive energy representation with $\mathcal{H}(0)\cong\mathcal{H}'(0)$ by a unitary isomorphism $v\mapsto v'$, then $U(Av)=Av'$ defines a unitary map of $\mathcal{H}$ onto $\mathcal{H}'$ intertwining the action of $\widehat{\mathfrak{sl}_2}$.

3. Character formulas for affine Lie algebras. The character of the spin j representation V_j of $SU(2)$, and hence $\mathfrak{sl}_2$, is given by

$$\chi_j(\zeta)=\frac{\zeta^{2j+1}-\zeta^{-2j-1}}{\zeta-\zeta^{-1}},$$

writing ζ for the element $\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}$ of $SU(2)$. The character of the level 1 spin j representation $\mathcal{H}_j$ ($j=0,1/2$) is given by

$$X_j(\zeta,q)=\mathrm{ch}\,\mathcal{H}_j=\sum_{k=0}^{\infty}\chi_j(\zeta)(q^{(j+k)^2}-q^{(j+k+1)^2})\varphi(q)=\sum_{n\in j+\mathbb{Z}}\zeta^{2n}q^{n^2}\varphi(q),$$

where

$$\varphi(q)=\prod_{n\geq 1}(1-q^n)^{-1}.$$

This formula can be deduced directly from first prinicples or by decomposing the fermionic representation on $\mathcal{F}^{\otimes 2}$. In both cases we use the fact that there are just two level one representations of $\mathcal{L}\mathfrak{sl}_2$, $\mathcal{H}_j$ with $j=0,1/2$, $\mathcal{H}_0$ involving only integer spin representations of $SU(2)$ and $\mathcal{H}_{1/2}$ representations only non–integer spins. We use the fact that the operators $L_n^{SU(2)}$ are given by the construction of the Virasoro algebra associated with the bosonic system $(H(k))$.

Method I. Direct treatment. We consider first the It is generated by a lowest energy vector Ω with $L_0\Omega=0$. Let $\mathcal{H}[i]=\{\xi\in\mathcal{H}_0:H(0)\xi=i\xi\}$. Since Ω is cyclic, these eigenspaces are invariant under the Virasoro algebra and are non–zero only if i is an integer. We know that $\mathcal{H}[0]$ is non–zero. We claim that $\mathcal{H}[i]$ is non–zero for every $i\in\mathbb{Z}$. Indeed $E(-n)$, $F(n)$ and $X=\frac{1}{2}[E(-n),F(n)]$ give a copy of $\mathfrak{sl}_2$ with $X\Omega=-n\Omega$. Hence $\xi_n=E(-n)^n\Omega\neq 0$ is a non–zero vector in $H[n]$. Similarly using $E(n)$, $F(-n)$ and $Y=\frac{1}{2}[E(n),F(-n)]$ as the copy of $\mathfrak{sl}_2$, it follows that $\xi_{-n}=F(-n)^n\Omega\neq 0$ is a non–zero vector in $H[-n]$. Now $\mathcal{H}[0]$ has a unique lowest energy vector and is an irreducible oscillator module. For any singular vector would satisfy $L_0^{SU(2)}\xi=0$ and hence be a multiple of the vacuum vector.

We claim that the ξ_n is a lowest energy vector in $\mathcal{H}[n]$ and generates $\mathcal{H}[n]$ as an oscillator module. Clearly $[L_0,E(-n)^n]=n^2E(-n)^n$ and $[L_0,F(-n)^n]=n^2F(-n)^n$ so that $L_0\xi_n=n^2$. But with the present normalisation $L_0=H(0)^2+2\sum_{n>0}H(-n)H(n)$, so that $H(m)\xi_n=0$ for $m>0$. If there were other singular vectors in $\mathcal{H}[n]$ with $H(0)\eta=n\eta$ and $H(m)\eta=0$ for $m>0$, then $L_0\eta=n^2\eta$ so that $F(n)^n\eta$ or $E(n)^n\eta$ would be a non–zero vector in $\mathcal{H}[0]$. But then it would be be proportional to the vacuum vector Ω and, by $\mathfrak{sl}_2$ theory, η would be proportional to ξ_n. Consequently the character of the vacuum representation is $\sum_{n\in\mathbb{Z}}q^{n^2}z^n\varphi(q)$.

We repeat this argument for the spin 1/2 representation $\mathcal{H}_{1/2}$. In this case the lowest energy representation gives non-zero lowest energy vectors in $\xi_{\pm 1/2}=\mathcal{H}[\pm 1/2]$ of energy 1/4 with respect to the operator $L_0^{SU(2)}$. This time the vectors $\xi_{n+1/2}=E(-n-1)^n\xi_{1/2}$ and $\xi_{-n-1/2}=F(-n-1)^n\xi_{-1/2}$ are non–zero vectors in $H[\pm(n+1/2)]$. They have energy $(n+1)n+1/4=(n+1/2)^2$. As before an $\mathfrak{sl}_2$ argument shows that, up to scalar multiples, these are the only singular vectors in $\mathcal{H}[\pm(n+1/2)]$. Hence the character of the spin 1/2 representation is $\sum_{n\in 1/2+\mathbb{Z}}q^{n^2}z^n\varphi(q)$.

Method II. Treatment using fermions. Using the direct sum decomposition from the previous section, as a representation of $U(1)$, $\mathcal{F}$, as a representation of $U(1)$, has character

$$\theta(z,q)=\sum_{n\in\mathbb{Z}} z^n q^{n^2/2}\varphi(q),$$

for $z\in U(1)$. We now use the identity:

$$\begin{aligned}\theta(z,q)\theta(\zeta^{-1},q) &= \sum_{m,n\in\mathbb{Z}}(z\zeta^{-1})^m(z\zeta)^n q^{(m^2+n^2)/2}\varphi(q)^2\\ &= \sum_{j=0,1/2}\sum_{a,b\in j+\mathbb{Z}} z^{2a}\zeta^{2b}q^{a^2+b^2}\varphi(q)^2\\ &= \sum_{j=0,1/2} X_j(\zeta,q)\Psi_j(z,q),\end{aligned}$$

where

$$\Psi_j(z,q)=\sum_{m\in j+\mathbb{Z}} z^{2m}q^{m^2}\varphi(q).$$

We have already exhibited explicit singular vectors $\Omega_p\in\mathcal{F}$ with $p\in\mathbb{Z}$. By boson–fermion duality these exhaust the singular vectors. This yields singular vectors $\Omega_p\otimes\Omega_q$ in $\mathcal{F}^{\otimes 2}$. If we split up the character of $\mathcal{F}^{\otimes 2}$ as the sum of characters of integer and non–integer spin $\Theta_j(z,\zeta,q)$, then $\Theta_j(z,\zeta,q)=X_j'(\zeta,q)\cdot\Psi_j'(z,q)$, with $X_j'(\zeta,q)$ the character of $\mathcal{H}_j$ and $\Psi_j'(z,q)$ the character of the mutliplicity space of $\mathcal{H}_j$. If we fix a spin $j=0,1/2$, then there are certainly singular vectors of charge $k+j$ for each $k\in\mathbb{Z}$ for the $H(n)$'s. Thus

$$X_j'(\zeta,q)\geq X_j(\zeta,q)=\sum_{n\in j+\mathbb{Z}}\zeta^{2n}q^{n^2}\varphi(q), \tag{1}$$

where the inequality is to be taken coefficient by coefficient. Similarly, for the character of the multiplicity space,

$$\Psi_j'(z,q)\geq\Psi(z,q)=\sum_{n\in j+\mathbb{Z}} z^{2n}q^{n^2}\varphi(q). \tag{2}$$

On the other hand the identity above implies that

$$\sum_j X_j'(\zeta,q)\Psi_j'(z,q)=\sum X_j(\zeta,q)\Psi_j(z,q),$$

so that equality must hold in (1) and (2).

Remark. In Section 8 we will introduce shift operators which together with the $H(n)$'s and the $K(n)$'s act irreducibly on the $\mathcal{H}_j$'s and their multiplicity spaces. This gives a more conceptual operator–theoretic explanation of the second proof.

4. Existence of singular vectors in the oscillator representations. According to the character formula $\mathcal{H}_j$ has a series of vectors ξ_m ($m\in j+\mathbb{Z}$) with $H(0)\xi_m=m\xi_m$ and $H(n)\xi_m=0$ for $m>0$. Moreover $\mathcal{H}_j$ is the direct sum of the cyclic oscillator modules $\mathcal{K}_m$ generated by these vectors; note that, if $H=H(0)$, then $\mathcal{K}_m$ is the m–eigenspace of H. Following Graeme Segal, we construct non-zero singular vectors in $\mathcal{K}_j\subset\mathcal{H}_j$. In fact the vector ξ_m satisfies $H\xi_m=m\xi_m$. If $m\geq 0$ then $\eta=E\xi_m$ lies in $\mathcal{K}_{m+1}$ and satisfies $L_0\eta=\frac{1}{2}m^2\eta$. Since the lowest energy level in $\mathcal{K}_{m+1}$ is $\frac{1}{2}(m+1)^2$ it follows that $\eta=0$, i.e. $E\xi_m=0$. Similarly if $m\leq 0$, $F\xi_m=0$. Thus ξ_m generates an irreducible $\mathfrak{sl}_2$–representation of dimension $2|m|+1$. If $m>0$, then $F^{[m]}\xi_m$ is a non–zero singular vector in $\mathcal{K}_j$ of energy $m^2/2$. If $m<0$ then $E^{[m]+1}\xi_m$ is a non–zero singular vector in $\mathcal{K}_j$ of energy $m^2/2$. This proves:

Proposition. *The oscillator representation with $H(0)=jI$ has non–singular singular vectors for the Virasoro algebra with energy $(j+n)^2/2$ for $n\geq 0$. The non–zero singular vector constructed above in $\mathcal{K}_j$ of energy $m^2/2$ ($m\geq 0$) generates a copy of V_m as an $\mathfrak{sl}_2$-module.*

We shall call these singular vectors *Goldstone vectors*. In Section 8, following Segal, we will use vertex operators to deduce the explicit formulas of Goldstone for these vectors.

5. Uniqueness of singular vectors in the oscillator representations. Let (a_m) satisfy $[a_m, a_n] = m\delta_{m+n,0}I$. These act as operators on $V = \mathbb{C}[x_1, x_2, \ldots]$ via $a_0 = \sqrt{2}\mu$, $a_n = n\partial_{x_n}$ and $a_{-n} = x_n$ for $n > 0$. Set $L_0 = a_0^2/2 + \sum_{n>0} a_{-n}a_n$ and $L_k = \frac{1}{2}\sum_{p+q=k} a_p a_q$ for $k \neq 0$. Thus $[L_k, a_n] = -na_{n+k}$ and $[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}(m^3 - m)/12$.

Proposition. *For $n \geq 0$ let $V(n) = \{\xi : L_0\xi = (\mu^2 + n)\xi\}$ with $\mu \in \mathbb{R}$. Then $\dim V(n) = \mathcal{P}(n)$ the partition function and the subspace of singular vectors, i.e. sRolutions of $L_1\xi = 0 = L_2\xi$, is at most one-dimensional.*

Proof. If we define the degree of x_i to be i, then $V(\mu^2 + n)$ is spanned by all monomials of total degree n, so that $\dim V(n) = \mathcal{P}(n)$. We shall treat a more general case which evidently implies the proposition:

Lemma. *Let $W = \mathbb{C}[x_1, x_2, \ldots]$ with $\deg x_i = i$ and let $W(n)$ be the space spanned by monomials of total degree n, so thet $\dim W(n) = \mathcal{P}(n)$. For $a, b \in \mathbb{C}$ let*

$$A = a\partial_{x_1} + \sum_{n>0}(n+1)x_n\partial_{x_{n+1}},\ B = \frac{1}{2}\partial_{x_1}^2 + b\partial_{x_2} + \sum_{n\geq 1}(n+2)x_n\partial_{x_{n+2}}.$$

Then the space of solutions of $W_0(n) = \{p \in W(n) : Ap = 0,\ Bp = 0\}$ is at most one–dimensional.

Proof. For $i \geq 0$, let $W(n,i) = \{x_1^{n-i}q(x_2, \ldots) : \deg q = i\}$. Evidently

$$W(n) = \bigoplus_{i\geq 0} W(n,i)$$

so that any element $p \in W(n)$ has a decomposition $p = \sum p_i$ with $p_i \in W(n,i)$. Thus $p_i = x_1^{n-i}q_i(x_2, x_3, \ldots)$ with $\deg q_i = i$. We assume that the leading coefficient p_0 (a multiple of x_1^n) is zero and prove by induction on i that $q_i = 0$. Each q_i is a sum of linear combination of monomials $x^\alpha = x_2^{\alpha_2}\ldots x_n^{\alpha_n}$ with $\sum_{k\geq 2} k\alpha_k = i$. Suppose that $\alpha_2 > 0$. Then

$$Ax_1^{n-i}x^\alpha = a(n-i)x_1^{n-i-1}x^\alpha + 2x_1^{n-i+1}\alpha_2 x_2^{\alpha_2 - 1}x_3^{\alpha_3}\cdots + 3\alpha_3 x_1^{n-i}x_2^{\alpha_2+1}x_3^{\alpha_3-1}x_4^{\alpha_4}\cdots + \cdots.$$

Because of the inductive hypothesis, this is the only monomial which produces a term $x_1^{n-i+1}x_2^{\alpha_2-1}x_3^{\alpha_3}\cdots$ when A is applied. It follows that its coefficient must be zero. So only the coefficients of monomials $x_1^{n-i}x_3^{\alpha_3}x_4^{\alpha_4}\cdots$ The image under B of such a monomial is

$$Bx_1^{n-i}x_3^{\alpha_3}x_4^{\alpha_4}\cdots = \frac{1}{2}(n-i)(n-i-1)x_1^{n-i-2}x_3^{\alpha_3}x_4^{\alpha_4}\cdots + 3\alpha_3 x_1^{n-i+1}x_3^{\alpha_3-1}x_4^{\alpha_4}\cdots.$$

Because of the inductive hypothesis this is the only monomial which produces a term $x_1^{n-i-2}x_3^{\alpha_3}x_4^{\alpha_4}\cdots$ when B is applied. It follows that its coefficient must be zero. Thus $p_i = 0$ and hence $p = 0$, as required.

6. Density modules, primary fields and the Feigin–Fuchs product formula. For $\lambda, \mu \in \mathbb{C}$, we define the density module $V_{\lambda,\mu}$ to be the vector space with basis v_n ($n \in \mathbb{Z}$) and define operators ℓ_k on $V_{\lambda,\mu}$ by

$$\ell_k v_n = -(n + \lambda k + \mu)v_{n+k}.$$

It is easy to check that $[\ell_m, \ell_n] = (m-n)\ell_{n+m}$, so this defines a representation of the Virasoro algebra with $c = 0$. Clearly the change of basis by the shift operator $v_n \mapsto v_{n+k}$ gives a natural isomorphim between $V_{\lambda,\mu}$ and $V_{\lambda,\mu+k}$.

Recall that if $\pi : \mathfrak{g} \to \operatorname{End} W$ is a representation of a Lie algebra $\mathfrak{g}$, then the *dual representation* on the dual space W' is given by $(\pi'(X)\xi, v) = -(\xi, Xv)$ for $v \in W$. Let w_n be the canonical basis of $V_{1-\lambda,-\mu}$. We identify U with a subspace of the dual $V_{\lambda,\mu}$ via the pairing $(v_n, w_m) = \delta_{n+m,0}$. It is immediately verified that the dual representation restricts to the natural representation on $V_{1-\lambda,-\mu}$. With an obvious abuse of notation, we shall write $V_{1-\lambda,-\mu} \cong V'_{\lambda,\mu}$. These representations are the infinitesimal version of the action of

Diff S^1, or more properly a certain central extension, on the densities $e^{i\mu\theta}f(\theta)(d\theta)^\lambda$. The duality between $V_{\lambda,\mu}$ and $V_{1-\lambda,-\mu}$ is given by the pairing into 1–forms

$$(f(\theta)e^{i\mu\theta}(d\theta)^\lambda, g(\theta)e^{-i\mu\theta}(d\theta)^{1-\lambda}) \mapsto fgd\theta$$

followed by integration over the circle, i.e. $\frac{1}{2\pi}\int_0^{2\pi} fg\, d\theta$. Identifying v_n with $e^{in\theta}e^{i\mu\theta}(d\theta)^\lambda$, we see that

$$\ell_k(fe^{i\mu\theta}(d\theta)^\lambda)) = -e^{ik\theta}(i\frac{d}{d\theta} + k\lambda)(fe^{i\mu\theta})(d\theta)^\lambda.$$

Thus it is the action on $\mathbb{C}[e^{i\theta}, e^{-i\theta}]e^{i\mu\theta}$ given by the operators

$$\ell_k = -e^{ik\theta}(i\frac{d}{d\theta} + k\lambda).$$

Changing variables to $z = e^{i\theta}$, regarded as a formal variable, and introducing an extra factor z^λ, it may also be identified with the action on $\mathbb{C}[z, z^{-1}]z^{\mu+\lambda}$ given by

$$\ell_k = -z^{k+1}\frac{d}{dz} - (k+1)\lambda z^k.$$

For applications here λ and μ will be in $\mathbb{Z}$ (for applications to fusion we require $\lambda, \mu \in \frac{1}{4}\mathbb{Z}$).

For $i = 1, 2$, let $\mathcal{H}_i = L(1, h_i)$ with lowest energy vectors $\xi_i = \xi_{h_i}$. We define a primary field of type (λ, μ) between $\mathcal{H}_1$ and $\mathcal{H}_2$ to be a map $\Phi : \mathcal{H}_1 \otimes V_{\lambda,\mu} \to \mathcal{H}_2$ commuting with the action of the Virasoro algebra. Let $\Phi(n)\xi = \phi(\xi \otimes v_n)$. Then

$$[L_k, \Phi(n)] = -(n + k\lambda + \mu)\Phi(n + k).$$

We have the following uniqueness result:

Lemma A. *If $(\Phi(n)\xi_1, \xi_2) = 0$ then $\Phi(n) = 0$ for all $n \in \mathbb{Z}$. If Φ is non–zero, then $m = h_1 - h_2 - \mu$ is an integer and $\Phi(m)\xi_1$ is a non–zero multiple of ξ_2.*

Proof. Suppose that $(\Phi(n)\xi_1, \xi_2) = 0$. From the commutation relations it follows that for $n_i > 0$

$$(\Phi(n)L_{-n_k} \cdots L_{-n_1}\xi_1, \xi_2) = 0.$$

Hence $(\Phi(n)\xi, \xi_2) = 0$ for all ξ. From the commutation relations is then follows that for $n_i > 0$

$$(\Phi(n)\xi, L_{-n_k} \cdots L_{-n_1}\xi_2) = 0,$$

so that $(\Phi(n)\xi, \eta) = 0$ for all ξ, η and hence $\Phi(n) = 0$ for all $n \in \mathbb{Z}$. Finally if $(\Phi(m)\xi_1, \xi_2) \neq 0$, then $\Phi(m)\xi_1$ is a non–zero multiple of ξ_2. Hence $h_2 = h_1 - m - \mu$.

Remark. This result allows us to normalise a primary field by shifting numbering of the modes $\Phi(n)$ to $\Phi(n+k)$. Thus, if a primary field exist, we may assume that $\mu = h_1 - h_2$ and that $\Phi(0)\xi_1 = \alpha\xi_2$ with $\alpha \neq 0$.

In the language of vertex operators and vertex algebras, primary fields are usually described in terms of generating functions in a formal variable z. We set

$$\Phi(z) = \sum_{n\in\mathbb{Z}} \Phi(n)z^{-n-\delta}.$$

Then

$$[L_k, \Phi(z)] = z^{k+1}\frac{d\Phi(z)}{dz} + (k+1)\Delta z^k\Phi(z).$$

Taking coefficients of $z^{-n-\delta}$, we get

$$[L_k, \Phi(n)] = -(n+k+\delta-(k+1)\Delta)\Phi(n+k).$$

Thus $\lambda = -\Delta+1$ and $\mu = \delta-\Delta$. Hence $\Delta = 1-\lambda$ and $\delta = \mu+\lambda-1$.

Now the Verma module representation $M(1,j^2)$ $(j = 0, 1/2, 1, 3/2\ldots)$ certainly has one singular vector at energy $(j+1)^2 = j^2+d$ with $d = 2j+1$, since the representation $L(1,j^2)$ is a subrepresentation of the multiplicity space. This corresponds to a element of the universal enveloping algebra $\mathcal{U}_1$ generated by L_{-k} $(k \geq 1)$. Let $\mathcal{U}_k$ be the universal enveloping algebra generated by L_{-i} with $i \geq k$. It is the universal enveloping algebra of the Lie algebra generated by L_i for $i \geq k$. This is a Lie ideal in the Lie algebra generated by L_{-j} with $j \geq 0$. As a consequence every element of the algebra $\mathcal{U}_1$ can be written uniquely in the form $P = \sum_{k\geq 0} p_k L_{-1}^k$ and $\mathrm{ad}L_{-1}$ defines a derivation of $\mathcal{U}_2$. In particular the singular vector has the form Pv_{j^2} where

$$P \equiv P_d = \sum_{k=0}^{d} q_k L_{-1}^k.$$

Each term is a q_k is a sum of monomials $L_{-i_1}\cdots L_{-i_r}$ where $i_1+\cdots+i_r = d-k$. In particular q_d is a scalar. The following uniqueness result was stated by Fuchs.

Lemma B. *The coefficient of L_{-1}^d in P_d is non–zero; in particular if a singular vector exists, it is unique up to a scalar multiple.*

Proof. Suppose that $q_d = 0$. We prove that $P_d = 0$. Each q_d can be written as a linear combination of monomials $L_{-s}^{n_s}\cdots L_{-2}^{n_2}$ with $n_i \geq 0$. Amongst all monomials with non–zero coefficients, we can find one $p \geq 2$ minimal with $n_p > 0$. We can also choose this monomial so that $n = n_p$ is also minimal. Suppose that it occurs in q_c with $0 \leq c < d$. We may in addition assume that c is chosen to be maximal. Let $w = P_d v$ with v the cyclic lowest energy vector in Verma module. Then $L_{p-1}w = 0$. This can be written as

$$\sum_{k=0}^{c} L_{p-1}q_k L_{-1}^k v = \sum_{k=0}^{c}[L_{p-1}, q_k]L_{-1}^k v + q_k L_{p-1}L_{-1}^k v.$$

We look for terms ending with $L_{-p}^{n-1}L_{-1}^{c+1}$ in this expression. We first note that it follows by induction on k that if $j \geq 1$ then $L_j L_{-1}^k v$ lies in $\mathrm{lin}\,\{AL_{-1}^i v : i < k,\ A \in \mathcal{U}_2\}$. Indeed

$$L_j L_{-1}L_{-1}^{k-1}v = (j+1)L_{j-1}L_{-1}^{k-1}v + L_{-1}L_j L_{-1}^{k-1}v,$$

which has the same form since $L_{-1}\mathcal{U}_2 = \mathcal{U}_2 L_1$ and $L_{-1}^{k-1}v$ is an eigenvector of L_0. If $k \neq c$, then the minimality of p forces q_k to be in $\mathcal{U}_p$. So either $k > c$, in which case any monomial q_k could have a non–zero contribution from a monomial $L_{-s}^{m_s}\cdots L_{-p}^{m_p}$ with $m_p > n_p$. Clearly taking the Lie bracket with L_{p-1} can diminish the exponent of L_{-p} by at most one and at the same time must increase the exponent of L_{-1} by at least one. So no monomials ending with $L_{-p}^{n-1}L_{-1}^{c+1}$ can appear with non–zero coefficient. If $k < c$, there is no way to increase the power of L_{-1}^k to L_{-1}^{c+1} by taking the Lie bracket with L_{p-1}. For $k = c$ and the terms $[L_{p-1}, AL_{-p}^n]$ with A a monomial in $\mathcal{U}_{p+1}$, we have

$$[L_{p-1}, AL_{-p}^n] = [L_{p-1}, A]L_{-p}^n + A[L_{p-1}, L_{-p}^{n_p}].$$

The first term lies in $\mathcal{U}_2$ while for the second

$$[L_{p-1}, L_{-p}^n]v = (2p-1)\sum_{a+b=n-1} L_{-p}^a L_{-1}L_{-p}^b v = (2p-1)nL_{-p}^{n-1}L_{-1} + B,$$

where $B \in \mathcal{U}_2$. There could be several terms in q_c with $n_p = n$, but on bracketing with L_{p-1} and taking the term ending with $L_{-p}^{n-1}L_{-1}^{c+1}$ is equal to $AL_{-p}^{n-1}L_{-1}^{c+1}$. Thus all these terms are distinct. But then there can be no cancellation and the coefficient of $L_{-r}^{n_r}\cdots L_{-p}^{n_p-1}L_{-1}^{c+1}v$ must therefore be non–zero, a contradiction.

Remark. Fuchs used a slightly different ordering on monomials, which can also be used to give prove the uniqueness theorem. For a monomial $L_{-s}^{n_s}\cdots L_{-2}^{n_2}L_k$, we set $n_1 = d-k$ and use the lexicographic ordering determined by $(n_1, n_2, \ldots)$. The proof is almost word–for– word that used in the second proof of Fuchs' algorithm in Section 10.

Let $P_d \in \mathcal{U}_1$ be the element giving the singular vector normalized so that the coefficient of L_{-1}^d equals 1. In the representation $V_{\lambda,\mu}$, we evidently have

$$P_d v_0 = a_d(\lambda,\mu) v_{-d}$$

where $a_d(\lambda,\mu)$ is a inhomogenous polynomial of degree d in λ and μ. Indeed since the L_{-1}^d appears with coefficient 1, the formula for the action shows that the coefficient of μ^d is $(-1)^d$. Using the coset construction in Section 8 and an explicit formula for P_d in Appendix C, we give two methods to establish the following product formula of Feigin–Fuchs, which we state first in the two simplest cases:

Feigin–Fuchs product formula. *Let $j = 0, 1/2, 1, 3/2, \ldots$ and set $S = \{-j, -j+1, -j+2, \ldots, j\}$.*
(a) $a_d(0,\mu) = (-1)^d \prod_{k\in S}(\mu + j^2 - k^2)$.
(b) $a_d(1,\mu) = (-1)^d \prod_{k\in S}(\mu + j^2 - (k+1)^2)$
(c) $a_d(p^2,\mu) = (-1)^d \prod_{k\in S}(\mu + j^2 - (k+p)^2)$ for $p \geq 0$.
(d) $a_d(\lambda,\mu)^2 = \prod_{k\in S}((\lambda - \mu - j^2 + k^2)^2 - 4\lambda k^2)$.

We have the following necessary condition for the existence of a primary field:

Lemma C. *If there is a non–zero primary field from $L(1,j^2)$ to $L(1,h)$ of type (λ,μ), then*

$$a_d(1-\lambda, h-j^2) = 0.$$

Proof. Let Φ be a non–zero primary field of type (λ,μ) from $L(1,j^2)$ to $L(1,h)$. Then we may assume that Φ is normalised so that $\mu = j^2 - h$ and $\Phi(0)\xi_{j^2} = \alpha\xi_h$ for some constant $\alpha \neq 0$. Let $\xi = \xi_{j^2}$ and $\eta = \xi_h$. Then $(\Phi(0)\xi, \eta) \neq 0$. On the other hand $P_d\xi = 0$. Let $P_d = \sum a_m L_{-1}^{m_1} L_{-2}^{m_2}\cdots$. For an operator we set $\pi(L_j)A = [L_j, A]$. This defines an action of the Virasoro algebra with $c = 0$. Now

$$0 = (\Phi(d)P_d\xi, \eta) = \sum a_m (\pi(-L_{-n})^{m_n}\cdots \pi(-L_{-2})^{m_2}\cdot \pi(-L_{-1})^{m_1})\cdot \Phi(d)\xi, \eta).$$

On the other hand

$$\sum a_m \pi(-L_{-n})^{m_n}\cdots \pi(-L_{-2})^{m_2}\cdot \pi(-L_{-1})^{m_1})\cdot \Phi(d) = b\Phi(0),$$

where for the basis (v_i) of $V_{\lambda,\mu}$

$$\sum a_m(-L_{-n})^{m_n}\cdots (-L_{-2})^{m_2}\cdot (-L_{-1})^{m_1})v_d = bv_0.$$

Thus we require that b should vanish. To calculate b, we take (w_j) to be a dual basis of $V_{1-\lambda,-\mu}$. Then

$$b = \sum a_m((-L_{-n})^{m_n}\cdots (-L_{-2})^{m_2}(-L_{-1})^{m_1}v_d, w_0) = (v_d, P_d w_0).$$

Thus $P_d w_0 = bw_{-d}$ in $V_{1-\lambda,-\mu}$ and hence

$$b = a_d(1-\lambda,-\mu) = a_d(1-\lambda, h-j^2).$$

Remark. This result can be seen a little more transparently if the dual of $V_{\lambda,\mu}$ is used. Indeed, passing to a suitable completion of the tensor product, a primary field of type (λ,μ) from $L(c,h_1)$ to $L(c,h_2)$ can be regarded as a map $\mathcal{H}_1 \to \mathcal{H}_2 \widehat{\otimes} V_{\lambda',\mu'}$, where $\lambda' = 1-\lambda$ and $\mu' = -\mu$. Explicitly the map becomes

$$\xi \mapsto \Phi(z)\xi = \sum \varphi(n)\xi z^{-n},$$

where z^k here is identified with a basis element of $V_{\lambda',\mu'}$. If $v_{h_2} = \phi(-k)v_{h_1}$, then, picking out the coefficient of v_{h_2}, the condition $P(L_{-1}, L_{-2}, \ldots)v_{h_1} = 0$ evidently implies that $P(\ell_{-1}, \ell_{-2}, \ldots)z^k = 0$, i.e. the condition stated in Lemma C. It is therefore natural to refer to the primary field as having *dual type* (λ', μ'). Thus if the primary field is normalised so that $\Phi(0)\xi_1 = \alpha\xi_2$ with $\alpha \neq 0$, then $\mu' = h - j^2$ and Lemma C requires that $(a_d(\lambda', \mu') = 0$.

7. Multiplicity one theorem. We wish to prove the following theorem:

Theorem A. *The action of the Virasoro algebra on the multiplicity space M_m of V_m ($m \geq 0$, $m \in j + \mathbb{Z}$) in $\mathcal{H}_j$ is irreducible for $j = 0, 1/2$.*

Let us start by showing that it suffices to prove:

Theorem B. *The Goldstone vectors are the only singular vectors in $\mathcal{K}_m = \{\xi \in \mathcal{H}_j : H\xi = m\xi\}$ for $m \in j + \mathbb{Z}$.*

Proof that Theorem B implies Theorem A. The multiplicity space M_m can be identified with

$$M_m = \{\xi \in \mathcal{H}_j : H\xi = m\xi,\, E\xi = 0\} \subset \mathcal{K}_m = \{\xi : H\xi = m\xi\}.$$

If the action on M_m is not irreducible, then M_m would have to contain a singular vector of higher energy. But by Theorem B any such vector would have to be a Goldstone vector in $\mathcal{K}_m$. But, by the proposition in Section 4, ξ_m is the only Goldstone vector in $\mathcal{K}_m$ annihilated by E.

Thus we have only to prove Theorem B. By the uniqueness theorem it suffices to show that there are non singular vectors in $\mathcal{K}_j$ with energy not of the form $m^2/2$ with $m \in j + \mathbb{Z}$. For this we require the notion of a particularly simple kind of primary field. Let $V_{0,0} = \mathbb{C}[z, z^{-1}]$ with L_n acting as the operator $\ell_n = -z^{n+1}d/dz$. A$(0,0)$–primary field is an operator

$$\Phi : L(1, h_1) \otimes V_{0,0} \to L(1, h_2)$$

which intertwines the action of the Virasoro algebra.

Lemma A. *Let $\Phi(n)\xi = \Phi(\xi \otimes z^n)$ for $\xi \in L(1, h_1)$. Then $[L_m, \Phi(n)] = -n\Phi(n+m)$. Moreover any such assignment corresponds to a $(0,0)$–primary field via the above formula.*

Proof. Equivariance implies that

$$L_m\Phi(n)\xi = L_m(\Phi(\xi \otimes z^n)) = \Phi(L_m\xi \otimes z^n) + \Phi(\xi \otimes \ell_m z^n) = \Phi(n)L_m\xi - m\Phi(x, m+n).$$

Lemma B. *If $(\Phi(n))$ is a $(0,0)$–primary field so is $\Psi(n) = \Phi(-n)^*$.*

Proof. We have

$$[L_m, \Psi(n)] = [L_m, \Phi(-n)^*] = -[L_{-m}, \Phi(-n)]^* = -m\Phi(-n-m)^* = -m\Psi(n+m).$$

Lemma C. *If $(\Phi(n)\xi_1, \xi_2) = 0$ with ξ_i the lowest energy vector in $L(1, h_i)$, then $\Phi(n) = 0$ for all $n \in \mathbb{Z}$.*

Proof. Suppose that $(\Phi(n)\xi_1, \xi_2) = 0$. From the commutation relations it follows that for $n_i > 0$

$$(\Phi(n)L_{-n_k} \cdots L_{-n_1}\xi_1, \xi_2) = 0.$$

Hence $(\Phi(n)\xi, \xi_2) = 0$ for all ξ. From the commutation relations is then follows that for $n_i > 0$

$$(\Phi(n)\xi, L_{-n_k} \cdots L_{-n_1}\xi_2) = 0,$$

so that $(\Phi(n)\xi, \eta) = 0$ for all ξ, η and hence $\Phi(n) = 0$ for all $n \in \mathbb{Z}$.

Remark. Note that if $L_0\xi_i = h_i\xi_i$, then such a Φ can exist only if $h_1 - h_2$ is an integer.

Proposition. *If j is a non–negative half–integer and $k \neq (j+m)^2$ with $m \in \mathbb{Z}$ there are no non–zero $(0,0)$–primary fields from $L(1,j^2)$ to $L(1,k)$ or from $L(1,k)$ to $L(1,j^2)$.*

Proof. Using adjoints it suffices to prove the first assertion. If a non–zero primary field exists, then by Lemma C of Section 6, necessarily $a_d(1,j^2-k)$ would have to vanish. On the other hand by the Feigin–Fuchs product formula, if $S=\{-j,-j+1,\cdots,j-1,j\}$, we know that

$$a_d(1,j^2-k)=\prod_{t\in S}((t-1)^2-k)\neq 0.$$

Proof of Theorem B. We have to prove that the Goldstone vectors are up to scalar multiples the only singular vectors in $\mathcal{K}_m$. Let P be the projection onto the submodule generated by the Goldstone vectors in $\mathcal{K}_m$, and P_0 the projection onto the submodule generated by a Goldstone vector ξ. If there is another singular vector η not proportional to a Goldstone vector then its energy does not have the form $(j+k)^2$ with k an integer, by the uniqueness theorem. Let Q be be the projection on submodule generated by η. Then $\Phi(n)=QH(n)P_0$ an $P_0\Phi(n)Q$ are $(0,0)$–primary fields since L_m commutes with P_0 and Q and $[L_m,H(n)]=-nH(n+m)$. By the proposition, we must have $QH(n)P_0=0$ and $P_0H(n)Q=0$. Hence it follows that $QH(n)P=0=PH(n)Q$. But then $H(n)$ leaves the subspace $P\mathcal{K}_m$ invariant. Since the $H(n)$'s act irreducibly on $\mathcal{K}_m$ it follows that $P=I$, a contradiction. So there are no singular vectors other than the Goldstone vectors.

Corollary 1. *The character of $L(1,m^2)$ for m a non–negative half–integer is $(q^{m^2}-q^{(m+1)^2})\varphi(q)$.*

Corollary 2. *Up to a scalar multiple $P_m\cdot\xi_m$ is the unique singular vector of energy $(m+1)^2$ in $M(1,m^2)$ and the quotient module is irreducible. The other singular vectors in $M(1,m^2)$ are given by $P_{m+k-1}\cdots P_{m+1}P_m\xi_m$ (of energy $(m+k)^2$).*

Proof. By the character formula, there is only one singular vector η of energy $(m+1)^2$ in $M=M(1,m^2)$ and no singular vectors of lower energy except the generating vector of energy m^2. The submodule N generated by is isomorphic to $M(1,(m+1)^2)$. By the character formula $M(1,m^2)/N$ is irreducible. Any singular vector in M not in N its image would give a non–zero singular vector in $L(1,m^2)=M/N$ and would have to be a multiple of the generating vector. So it follows that all singular vectors of energy greater than or equal to $(m+1)^2$ lie in N, whence the result.

8. Proof of the Feigin–Fuchs product formula using primary fields. We shall prove below that if j, j_1, j_2 are integers then there is a primary field of dual type $(j_1^2, j_2^2-j^2)$ from $L(1,j^2)$ to $L(1,j_2^2)$ provided that V_{j_2} occurs as a component in $V_j\otimes V_{j_2}$. Thus $a_d(j_1^2,j_2^2-j^2)=0$, by Lemma C in Section 6. This is enough information to compute $a_d(\lambda,\mu)$ completely.

We fix j, with $d=2j+1$, and freeze j_2, and hence $\mu=j_2^2-j^2$, with $j_2\geq j$. Then the possible values of j_1 are j_2+k with $k\in S=\{-j,-j+1,\cdots,j-1,j\}$. But $a_d(\lambda,j_2^2-j^2)$ has degree at most d in λ and $d=|S|$. Hence

$$a_d=a_d(\lambda,j_2^2-j^2)=C\prod_{k\in S}(\lambda-(j_2+k)^2),$$

where the constant C might depend on μ. But then, using the fact that $-S=S$, we can write

$$\begin{aligned}a_d(\lambda,j_2^2-j^2)^2&=C^2\prod_{k\in S}(\lambda-j_2^2-k^2-2j_2k)(\lambda-j^2-k^2+2j_2k)\\&=C^2\prod_{k\in S}((\lambda-j_2^2-k^2)^2-4j_2^2k^2)\\&=C^2\prod_{k\in S}((\lambda-\mu-j^2-k^2)^2-4(j^2+\mu)k^2).\end{aligned}$$

Since μ can take infinitely many values, it follows that $C^2=1$ and

$$a_d(\lambda,\mu)^2=\prod_{k\in S}((\lambda-\mu-j^2-k^2)^2-4(j^2+\mu)k^2).$$

On the other hand

$$\begin{aligned}(\lambda-\mu-j^2-k^2)^2-4(j^2+\mu)k^2 &= (\lambda+\mu)^2-2(\lambda-\mu)(j^2+k^2)+(j^2+k^2)^2-4j^2k^2-4\mu k^2\\ &= \lambda^2-2\lambda\mu+\mu^2-2\lambda(j^2+k^2)+2\mu(j^2-k^2)+(j^2-k^2)^2\\ &= (-\mu-j^2+k^2)^2+2\lambda(-\mu-j^2+k^2)+\lambda^2-4k^2\lambda\\ &= (\lambda-\mu-j^2+k^2)^2-4k^2\lambda.\end{aligned}$$

Thus we have

$$a_d(\lambda,\mu)^2=\prod_{k\in S}[(\lambda-\mu-j^2+k^2)^2-4k^2\lambda].$$

When $k=p^2$, this can be rewritten

$$a_d(p^2,\mu)^2=\prod_{k\in S}(-\mu-j^2+(k+p)^2)^2,$$

again using the fact that $S=-S$. Since the coefficient of μ^d in $a_d(\lambda,\mu)$ is $(-1)^d$, we deduce that

$$a_d(p^2,\mu)=(-1)^d\prod_{k\in S}(\mu+j^2-(k+p)^2),$$

as required.

We now explicitly construct all these primary fields. First we recall a version of the vertex operator formulation of the boson–fermion correspondence. We define the shift operator U on fermionic Fock space $\mathcal{F}$ by

$$U(e_{i_1}\wedge e_{i_2}\wedge e_{i3}\wedge\cdots)=e_{i_1+1}\wedge e_{i_2+1}\wedge e_{i_3+1}\wedge\cdots$$

By definition U is unitary and $Ue_iU^*=e_{i+1}$. Thus uniquely specifies U up to a scalar multiple, because if U' is another such unitary, then $V=U'U^*$ commutes with the e_i's and their adjoints. Since the vacuum vector is characterised up to a scalar multiple by the conditions $e_i\Omega=0$ for $i\geq 0$ and $e_i^*\Omega=0$ for $i<0$, $V\Omega=\zeta\Omega$ for some $\zeta\in\mathbb{T}$. Since Ω is cyclic for the e_i's and their adjoints, it follows that $V=\zeta I$. The shift operator has the following properties, which show in particular that it defines by conjugation an automorphism of $\mathfrak{a}$:

Lemma A. *(a)* $Ue_iU^*=e_{i+1}$
(b) $UL_kU^*=L_k+a_k+\frac{1}{2}\delta_{k,0}I$
(c) $Ua_nU^*=a_n+\delta_{n,0}I$
(d) The system $\mathfrak{a}$, U *with the relations* $Ua_nU^*=a_n+\delta_{n,0}c, UdU^*=d+a_0+\frac{1}{2}c$ *and* $d=a_0^2/2+\sum_{n>0}a_{-n}a_n$ *has a unique irreducible positive energy representation with* c *acting as the scalar* I *and* a_0 *having integer eigenvalues.*
(e) If m *is a positive integer, the system* $\mathfrak{a}$, V *with the relations* $Va_nV^*=a_n+m\delta_{n,0}c$, $VdV^*=d+ma_0+\frac{m^2}{2}c$ *and* $d=a_0^2/2+\sum_{n>0}a_{-n}a_n$ *has* m *inequivalent irreducible positive energy representations with* c *acting as the scalar* I *and* a_0 *having integer eigenvalues. These representations can be characterised as having a lowest energy vector of charge* i *and energy* $i^2/2$ *with* $i=0,\ldots,m-1$*. For each* i*, the other eigenvalues of* a_0 *are congruent to* i *modulo* m*.*

Proof. (a) has already been proven. For (b), note that

$$[L_k-UL_kU^*,e_i]=-(i+(k+1)/2)e_{i+k}+(i+(k-1)/2)e_{i+k}=-e_{i+k}=[-a_k,e_i].$$

Hence $UL_kU^*-L_k-a_k$ commutes with the e_i's; the same relation for $-k$ in place of k, shows that it also commutes with their adjoints. Thus it carries Ω onto a scalar multiple of itself and by cyclicity is equal to mutliplication by this scalar. The scalar need only be computed for $k\geq 0$, by hermiticity. When $k>0$ we get 0 while for $k=0$, we get $UL_0U^*=L_0+a_0+1/2$. For (c), note that $Ua_nU^*-a_n$ commutes with the e_i's and their adjoints so is a scalar operator, which only need be calculated for $n\geq 0$ by looking at how it acts on Ω. For $n>0$ we get 0, while for $n=0$, we get $Ua_0U^*-a_0=I$. Finally to prove (d), in

the inner product space $\mathcal{H}$ take a vector ξ_0 of lowest energy which is an eigenvector of a_0. Thus $d\xi = h\xi$ and $a_0\xi = \mu\xi$. Then the vectors $\xi_n = U^{-n}\xi$ satisfy $a_0\xi_n = (\mu + n)\xi_n$ and $d\xi_n = \frac{1}{2}n^2\xi_n$. Each ξ_n generates an irreducible $\mathfrak{a}$–module, necessarily mutually orthogonal since the eigenvalues of a_0 are different on each. By irreducibility their direct sum gives the whole inner product space. But this decomposition shows that, after subtracting hI from d and μI from a_0, $\mathcal{H}$ can be written as a tensor product $\mathcal{K} \otimes E$ where $\mathcal{K}$ is the standard irreducible representation of (a_n) $(n \neq 0)$ and d and E is the standard inner product space with orthonormal basis e_n on which $Ue_n = e_{n+1}$, $de_n = \frac{1}{2}n^2\xi_n$ and $a_0e_n = ne_n$. To prove (e), note that taking $V = U^m$, the space $E = \mathbb{C}[\mathbb{Z}]$ splits up into m cosets for the subgroup $m\mathbb{Z}$ corresponding to the subgroup generated by V. Moreover each coset gives an irreducible summand with the vectors Ω_i $(i = 0, \ldots, m-1)$ giving the corresponding lowest energy vectors. The representations are inequivalent because i is specified as the smallest non–negative solution λ of $a_0v = \lambda v$, $L_0v = \frac{1}{2}\lambda^2v$.

Corollary. *In the setting of (e), the Virasoro operators assocyaed with the a_n's satisfy $VL_kV^* = L_k + ma_k + \frac{m^2}{2}\delta_{k,0}I$.*

Proof. It suffices to note that this holds for $V = U^m$.

Now for $m \in \mathbb{Z}$ we define the formal power series z and z^{-1}

$$\Phi_m(z) = U^{-m}z^{ma_0}\exp(\sum_{n>0}\frac{m \cdot z^n a_{-n}}{n})\exp(\sum_{n<0}\frac{m \cdot z^n a_{-n}}{n}) = \sum \Phi_m(n)z^{-n}.$$

If we define

$$E_\pm^m(z) = \exp(\sum_{\pm n>0}\frac{-m \cdot z^{-n}a_n}{n}),$$

then

$$\Phi_m(z) = U^{-m}z^{ma_0}E_-^m(z)E_+^m(z).$$

To see that this is well defined, note that, when applied to a finite energy vector ξ, this becomes a formal power series in z multiplied by a power of z^{-1}, since $E_+^m(z)\xi$ is a polynomial in z^{-1} with vector coefficients. We shall simply write $E_\pm(z)$ for $E_\pm^1(z)$. The notation has been chosen consistently so that, for $m > 0$,

$$E_\pm^m(z) = E_\pm(z)^m.$$

Fubini–Veneziano relations. $[a_i, \Phi_m(z)] = mz^i\Phi_m(z)$ *and* $[L_k, \Phi_m(z)] = z^{k+1}\Phi_m'(z) + \frac{m^2}{2}(k+1)z^k\Phi_m(z)$.

This result can be proved in a number of ways, the most conceptual using operator product expansions, which is now usually formulated within the language of vertex algebras. The original proof was along those lines and is described in **[14]**. Here we give a very elementary proof relying on the boson–fermion duality and no detailed computations. In Appendix A we give two other proofs of the relations: the first is a simple indirect proof based on a uniqueness result for primary fields associated with the system $(a_n), U, L_0$; the second is a direct verification by brute force along the lines sketched by Frenkel and Kac **[10]**.

Lemma B. *Let A be a formal power series in z (or z^{-1}) with operator coefficients and let D be an operator such that $[D, A]$ commutes with A. Then $[D, e^A] = [D, A]e^A$.*

Proof. We have $[D, A^N] = \sum_{p+q=N-1} A^p[D, A]A^q = N[D, A]A^{N-1}$.

Corollary. *(a)* $[a_n, E_+(z)] = 0 = [a_{-n}, E_-(z)]$ *if* $n \geq 0$.
(b) $[a_n, E_-(z)] = z^nE_-(z)$, $[a_{-n}, E_+(z) =]z^{-n}E_+(z)$ *if* $n > 0$.
(c) $[L_0, E_+(z)] = (\sum_{n>0} a_nz^{-n})E_+(z)$, $[L_0, E_-(z)] = (\sum_{n>0} a_{-n}z^{-n})E_-(z)$.

Proof. These formulas are straightforward consequences of the lemma, setting $D = a_n$ or L_0 and

$$A = \sum_{\pm n>0}\frac{m}{n}a_nz^{-n}.$$

Proof of the Fubuni–Veneziano relations. If $i \neq 0$, then a_i commutes with U and A_0, while

$$[a_i, E_-(z)E_+(z)] = mz^i E_-(z)E_+(z).$$

Hence $[a_i, \Phi_m(z)] = mz^i\Phi_m(z)$. For $i = 0$, we have $U^m a_0 U^{-m} = a_0 + mI$ and hence $[a_0, U^{-m}] = mU^{-m}$, so that $[a_0, \Phi_m(z)] = m\Phi_m(z)$, It follows that $\Phi_m(n)$ carries $\mathcal{F}[k]$ into $\mathcal{F}[k+m]$.

If $k = 0$, $UL_0U^* = L_0 + a_0 + \frac{1}{2}I$, so that it follows by induction that

$$U^m L_0 U^{-m} = L_0 + ma_0 + \frac{m^2}{2}I.$$

Premultiplying by U^{-m}, we get

$$[L_0, U^{-m}] = U^{-m}(ma_0 + \frac{m^2}{2}I).$$

Hence

$$[L_0, \Phi_m(z)] = U^{-m}z^{ma_0}(ma_0 + \frac{m^2}{2})E_-(z)E_+(z) + U^{-m}z^{ma_0}(B_-E_-E_+ + E_-B_+E_+),$$

where $B_\pm(z) = \sum_{\pm n>0} ma_n z^{-n}$. On the other hand

$$z\frac{d\Phi_m(z)}{dz} = U^{-m}(ma_0)z^{ma_0}E_-E_+ + U^m z^{-ma_0}(B_-E_-E_+ + E_-B_+E_+).$$

Thus

$$[L_0, \Phi_m(z)] = z\Phi_m'(z) + \frac{m^2}{2}\Phi_m(z).$$

We also have the relations

$$U\Phi_m(z)U^* = z^m\Phi_m(z), \;\; z^{-km}\Phi_m(z)\Omega_k|_{z=0} = \Omega_{k+m}.$$

The first is an immediate consequence of the relation $Ua_0U^* = a_0 + I$. The second follows from the first and the relation $\Phi_m(z)\Omega|_{z=0} = \Omega_m$.

We shall say that $\phi(n) : \mathcal{F} \to \mathcal{F}$ is a primary field for the system of operators (a_n), U and L_0 if

$$[a_n, \phi(k)] = m\phi(k+n), \;\; [L_0, \phi(n)] = -(n+\mu)\phi(n), \;\; U\phi(n)U^* = \phi(n+m),$$

for some $m \in \mathbb{Z}$ and $\mu \in \mathbb{R}$. In particular $\phi(n) = \Phi_m(n)$ satisfies these conditions with $\mu = -m^2/2$. Note that if $(\phi(n))$ is a primary field, so too is the shifted field $\psi(n) = \phi(n+k)$ for any $k \in \mathbb{Z}$. We now show that after a suitable shift $\phi(n)$ is proportional to the modes of the field $\Phi_m(z)$. Since U, a_n act cyclicly on Ω, if $\phi(n)\Omega = 0$, then ϕ is identically zero. Take n minimal subject to $\xi\phi(n)\Omega \neq 0$: this is possible because $\mathcal{F}$ is a positive energy representation. Shifting ϕ if necessary we many assume $n = 0$. Hence $a_i\xi = 0$ for all $i > 0$. Morevoer $a_0\xi = m\xi$. It follows that ξ is proportional to Ω_m. Scaling if necessary, we may assume that $\xi = \Omega_m$. Taking $k = n$ in $[L_0, \phi(k)] = -(k+\mu)\phi(k)$, we get $\mu = m - m^2/2$. We have the following uniqueness result for primary fields:

Lemma C. *(a) Let $\mathcal{K}_i$ be irreducible representations of $\mathfrak{a}$ with lowest energy vectors ξ_i with $a_0\xi_i = \lambda_i\xi_i$. Let $\phi(n) : \mathcal{K}_1 \to \mathcal{K}_2$ be operators such that $[d, \phi(n)] = -(n+\mu)\phi(n)$ and $[a_k, \phi(n)] = \nu\phi(n+k)$ for $\nu \in \mathbb{R}$ fixed. If $(\phi(n)\xi_1, \xi_2) = 0$ for all n, then $\phi(n) \equiv 0$.*
(b) Let (a_n), d, U be the bosonic system of operators acting on $\mathcal{F}$ and suppose that $\phi(n) : \mathcal{F} \to \mathcal{F}$ satisfies $[d, \phi(n)] = -(n+\mu)\phi(n)$, $[a_k, \phi(n)] = \nu\phi(n+k)$ and $U\phi(n)U^* = \phi(n+a)$ for some $\mu, \nu, a \in \mathbb{Z}$. Then if $\phi(n)\Omega$, we have $\phi(n) \equiv 0$. In particular if $(\phi(n)\Omega, \Omega_\nu) = 0$, then $\phi(n) \equiv 0$.
(c) Let (a_n), d, U be the bosonic system of operators acting on $\mathcal{F}$ and define $a_n^{(1)} = a_n \otimes I$, $a_n^{(2)} = I \otimes a_n$, $U_1 = U \otimes I$, $U_2 = I \otimes U$ on (the $\mathbb{Z}_2$–graded tensor product) $\mathcal{F}^{\otimes 2}$. Suppose that $\phi(n) : \mathcal{F}^{\otimes 2} \to \mathcal{F}^{\otimes 2}$ satisfies $[d, \phi(n)] = -(n+\mu)\phi(n)$, $[a_k^{(i)}, \phi(n)] = \nu_i\phi(n+k)$ and $U_i\phi(n)U_i^* = \phi(n+a)$ for some $\mu, \nu, a \in \mathbb{Z}$. Then if $\phi(n)\Omega \otimes \Omega = 0$, we have $\phi(n) \equiv 0$.

Proof. (a) From the commutation conditions it follows that $(\phi(n)p(a_{-1},a_{-2},\cdots))\xi_1,\xi_2)=0$ for any polynomial p. Thus $(\phi(n)\xi,\xi_2)=0$ for all ξ. But then by the commutation relations $(\phi(n)\xi,p(a_{-1},a_{-2},\cdots)\xi_2)=0$ for any polynomial p. But then $(\phi(n)\xi,\eta)=0$ for all ξ,η and hence $\phi(n)\equiv 0$.
(b) Note that by (a) it suffices to show that $(\phi(n)\Omega_r,\Omega_s)=0$ for all r,s, But then

$$(\phi(n)U^r\Omega,U^s\Omega)=(\phi(n-ar)\Omega,U^{s-r}\Omega)=(\phi(n-ar)\Omega,\Omega_{r-s})=0.$$

Finally $\phi(n):\mathcal{F}_0\to\mathcal{F}_\nu$, so by (b), the second condition implies that $\Phi(n)\Omega=0$.
(c) Let Ω' be an eigenvector of d in $\mathcal{F}$ and define $\psi(n)$ by

$$(\psi(n)\xi,\eta)=(\phi(n)\Omega\otimes\xi,\Omega'\otimes\eta).$$

Then by $\psi(n)$ satisfies the conditions in (b), so $\psi(n)=0$. Since Ω' was arbitrary, it follows that $\phi(n)\Omega\otimes\xi=0$ for all ξ. Fixing ξ_2,η_2 and defining $\psi^(n)$ by

$$(\psi'(n)\xi,\eta)=(\phi(n)\xi\otimes\xi_2,\eta\otimes\eta_2),$$

we deduce that $\psi'(n)=0$. Hence $\phi(n)\equiv 0$.

This uniqueness result allows us to prove the boson–fermion correspondence in its vertex operator formulation.

Example 1. $\Phi_1(z)=\sum e_n z^{-n-1}$ and $\Phi_{-1}(z)=\sum e_{-n}^* z^{-n}$. Indeed both sides of these identities satisfy the same covariance relations with $\Phi_1(-1)\Omega=\Omega_1=e_{-1}\Omega$ and $\Phi_{-1}(0)\Omega=\Omega_{-1}=e_0^*\Omega$.

As a consequence we get

$$[L_n,\Phi_{\pm1}(z)]=z^{n+1}\Phi{\pm_1}'(z)+\frac{1}{2}(n+1)z^n\Phi_{\pm1}(z). \qquad (*)$$

We claim that this implies the Fubini–Veneziano relations for $\Phi_{\pm m}(z)$. Indeed $(*)$ immediately implies the relations for the m^2–fold $\mathbb{Z}_2$–graded tensor product

$$\Theta_\pm(z)=\Phi_{\pm1}(z)\otimes\Phi_{\pm1}(z)\otimes\cdots\otimes\Phi_{\pm1}(z)$$

on $\mathcal{F}^{\otimes m^2}$. But this is just the vertex operator constructed with the operators

$$A_n=a_n\otimes I\otimes I\otimes\cdots\otimes I+\cdots+I\otimes\cdots\otimes I\otimes a_n$$

and

$$V=U\otimes U\otimes\cdots\otimes U$$

on $\mathcal{F}^{\otimes m^2}$. Now

$$[A_p,A_q]=m^2p\delta_{p+q,0}I,\quad VA_0V^*=A_0+m^2\cdot I.$$

Thus letting $a_n'=m^{-1}A_n$, we have

$$[a_p',a_q']=p\delta_{p+q,0}I,\quad Va_0'V^*=a_0'+m\cdot I.$$

The operators L_k on $\mathcal{F}^{\otimes m^2}$ satisfy

$$[L_k,a_n']=-na_{n+k}',\quad VL_kV^*=L_k+ma_k'+\frac{m^2}{2}\delta_{k,0}I.\ \ [L_k,\Theta_\pm(z)]=z^{k+1}\frac{d}{dz}\Theta_\pm(z)+\frac{m^2}{2}(k+1)z^k\Theta_\pm(z).$$

The first two relations are also true for the Virasoro operators L_k' constructed from the a_n''s, by the corollary to Lemma A. Thus L_k-L_k' commutes with a_n' and V, and therefore the third idenity is valid with L_k' replacing L_k. This holds on any irreducible submodule for the a_n''s and V, in particular those on which a_0' has integer eigenvalues. It is easy to see that the eigenvalues of A_0 run through all possible integers, so that

those of $a_0' = m^{-1}A_0$ in particular include all integers. Thus the Fubini–Veneziano relations are satisfied on any irreducible representation of the type discussed in part (e) of Lemma A. But these are exactly the irreducible representations for U^m and $\mathfrak{a}$ that occur in $\mathcal{F}$, so the Fubini–Veneziano relations hold on $\mathcal{F}$.

We now pass to $\mathcal{F}_2 = \mathcal{F}\otimes\mathcal{F}$. The unitary operators $V = U\otimes U^*$ and $W = U\otimes U$ act on this space. Now that V commutes with the action of $d_n = a_n\otimes I + I\otimes a_n$ and therefore leaves its mutliplicity space invariant, since these can be indentified with its space of singular vectors as explained above. More explicitly invariant under $b_n = a_n\otimes I - I\otimes a_n$ is invariant under V. Similarly any module invariant under b_n is invariant under W.

As a consequence the two level one representations of $\widehat{\mathfrak{sl}_2}$ are invariant under the vertex operators

$$\Psi_m(z) = \Phi_m(z)\otimes\Phi_{-m}(z) = V^{-m}z^{mb_0}\exp(\sum_{n>0}\frac{z^n mb_{-n}}{n})\exp(\sum_{n<0}\frac{mz^n b_{-n}}{n}) = \sum \Psi_m(n)z^{-n}.$$

The two systems (b_n), V and (d_n), W are examples of systems (A_n), U

$$[A_m, A_n] = 2m\delta_{m+n,0}\cdot I,\quad UA_mU^* = A_m + 2\delta_{m,0}I.$$

Thus if $L_0 = \frac{1}{2}A_0^2 + \sum_{n>0}A_{-n}A_n$, then we have

$$UL_0U^* - L_0 = \frac{1}{2}[(A_0+2I)^2 - A_0^2) = A_0 + 2I.$$

Such a system has two inequivalent positive energy representations with A_0 acting with integer eigenvalues, either all odd or all even. Thus both sets of operators $H(n), V$ and $K(n), w$ have only two irreducible representations with $H(0)$ and $K(0)$ having half integer eigenvalues, In one the eigenvalues run through the whole of $\mathbb{Z}$, and in the other through $\frac{1}{2}+\mathbb{Z}$. It follows that operators $H(n), V$ act irreducibly on each $\mathcal{H}_j$ and that the $K(n), W$ act irreducibly on the associated multiplicity spaces. The operator V also has a compatibility with the operators $E(n)$ and $F(n)$ on $\mathcal{F}^{\otimes 2}$:

$$VE(n)V^* = E(n+2),\quad VF(n)V^* = F(n-2).$$

In fact more generally

$$U_1E(n)U_1^* = E(n+1),\ U_2E(n)U_2^* = E(n-1),\ U_1F(n)U_1^* = F(n-1),\ U_2F(n)U_2^* = F(n+1).$$

Indeed it suffices to show the result for $E(n)$ since $E(n)^* = F(-n)$. The differences $X = U_1E(n)U_1^* - E(n+1)$ and $Y = U_2E(n)U_2^* - E(n-1)$ commute with the fermionic operators v_i and v_i^*, so that $X(\Omega\otimes\Omega)$ and $Y(\Omega\otimes\Omega)$ are proportional to $\Omega\otimes\Omega$. Hence X and Y are scalar operators. Since $[H,X] = X$ and $[H,Y] = Y$, we must have $X = 0 = Y$.

Example 2 (Frenkel–Kac–Segal). $E(z) = \Psi_1(z)$ and $F(z) = \Psi_{-1}(z)$ on $\mathcal{F}^{\otimes 2}$ and hence on the representations H_0 and $H_{1/2}$. Indeed the result follows from part (c) of Lemma C since, as we have just checked, both sides of these identities satisfy the same covariance relations and

$$\Psi_1(0)(\Omega\otimes\Omega) = \Omega_1\otimes\Omega_{-1} = E(-1)(\Omega\otimes\Omega),\quad \Psi_{-1}(0)(\Omega\otimes\Omega) = \Omega_{-1}\otimes\Omega_1 = F(-1)(\Omega\otimes\Omega).$$

Graeme Segal found explicit formulas for the Goldstone vectors in the vacuum representation $\mathcal{H}_0$. The same method can be applied in representation $\mathcal{H}_{1/2}$. Let $x_i = b_{-i}$ and define operators c_n ($n>0$) by the formal power series expansion

$$\Psi_-(z) = \exp\sum_{n>0}\frac{b_{-n}}{n}z^n = \sum_{n\geq 0}c_nz^n.$$

Thus $c_0 = 1$. For $n<0$, we define $c_n = 0$. Similarly we can define

$$\Psi_+(z) = \exp\sum_{n<0}\frac{b_{-n}}{n}z^n,$$

so that

$$\Psi_m(z) = V^{-m} z^{mb_0} \Psi_-(z)^m \Psi_+(z)^m.$$

Evidently

$$\Psi_+(z) = E_+(z) \otimes E_+(z)^{-1}, \quad \Psi_-(z) = E_-(z) \otimes E_-(z)^{-1}.$$

For any signature $f_1 \geq f_2 \geq f_3 \geq 0$, we define

$$X_f = \det c_{f_i - i + j}.$$

Remark. The proofs below can be understood in terms of the combinatorics of the Weyl character formula for $U(n)$. The characters of the irreducible representations π_f of $U(n)$ are indexed by signatures $f_1 \geq f_2 \geq \cdots f_n$ with the character given by $\chi_f(g) = \operatorname{Tr} \pi_f)(g) = \det z_i^{f_j + n - j} / \det z_i^{n-j}$ if $z_1, z_2, \ldots, z_n$ are the eigenvalues of g. The denominator is the Vandermonde determinant given by $\Delta(z) = \prod_{i<j}(z_i - z_j)$. The group $U(n)$ acts irreducibly on $S^k\mathbb{C}^n$: it is the representation with signature $(k, 0, 0, \ldots, 0)$. The group $U(m) \times U(n)$ acts on $S^k(\otimes C^m \otimes \otimes C^n)$ and the character if this representation on an element (g, h) is given by $\sum_{f_n \geq 0, |f| = k} \chi_f(g)\chi_f(h)$, where $|f| = f_1 + \cdots + f_n$. From this it follows that $\chi_f(g) = \det C_{f_i - i + j}$ (the Jacobi-Trudy identity) if $C_k = \chi_{(k,0,\ldots,0)}(g)$. Moreover the following generating function identity holds:

$$\prod_{i=1}^{n} f(z_i) = \sum_{f_n \geq 0} \chi_f(z) \cdot \det c_{f_i - i + j},$$

where $f(z) = 1 + c_1 z + c_2 z^2 + \cdots$ (with $c_0 = 1$ and $c_i = 0$ for $i < 0$). Note that the generating function for the X_j's giving the characters has $X_j = \sum z_i^j$.

Lemma D. *If A and B are formal power series of operators in z and w^{-1} such that $C = [A, B]$ is a multiple of the identity operator, then $e^A e^B = e^C e^B e^A$.*

Proof. By Lemma B, $[e^A, B] = Ce^A$ so that $e^A B e^{-A} = B + C$. Hence

$$e^A e^B e^{-A} = \exp e^A B e^{-A} = e^{B+C} = e^C e^B.$$

Corollary. *(a) $E_+(z)E_-(w) = (1 - w/z)^{-1} E_-(w) E_+(z)$.*
(b) $E_+(z)^{-1} E_-(w)^{-1} = (1 - w/z)^{-1} E_-(w)^{-1} E_+(z)^{-1}$.

Proof. To prove (a), let $A = \sum_{n>0} a_n z^{-n}/n$ and $B = \sum_{n<0} a_n w^{-n}/n$. Then

$$C = [A, B] = \sum_{n>0} z^{-n} w^{-n}/n = -\log(1 - \frac{w}{z}),$$

so that

$$e^C = \exp - \log(1 - \frac{w}{z}) = (1 - \frac{w}{z})^{-1}.$$

(b) can be proved similarly or by taking formal inverses in (a).

We now establish the formulas for Goldstone's vectors. As in Section 4, where the Goldstone vectors are defined, let ξ_j be a lowest energy unit vector in $\mathcal{H}_j$ with $H\xi_j = j\xi_j$. Then for $m \in \mathbb{Z}$ we may take $\xi_{m+j} = V^{-m}\xi_j$.

Lemma E (Goldstone's formulas). *If $k \geq 0$ is a half integer and $m \geq 0$ is an integer, the Goldstone vector in $\mathcal{H}_j[-k]$ of energy $(k+m)^2$ is proportional to $X_f \xi_{-k}$ where $f_1 = 2k + m, \ldots, f_m = 2k + m$ and $f_i = 0$ for $i > m$ and the Goldstone vector in $\mathcal{H}_j[k]$ of energy $(k+m)^2$ is proportional to $X_{f'}\xi_k$ where $f'_1 = m, \ldots, f'_{2k+m} = m$ and $f'_i = 0$ for $i > 2k + m$ (the transposed signature).*

Remark. In **[32]**, Segal only treats the case $j = 0$ and his statement is not quite correct since it omits the transpose. Note that the passage from $-k$ to k is equivalent to replacing the variables X_n by $-X_n$

(and hence inverting the generating series $\sum c_n z^n$). It corresponds to the action of the Weyl group element $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which establishes an isomorphism between $\mathcal{H}_j[-k]$ and $\mathcal{H}_j[k]$. Indeed the automorphism $b_n \mapsto -b_n$ fixes the Virasoro algebra generators and changes the sign of the eigenvalue of b_0.

Proof. By the corollary to Lemma D,

$$\Psi_+(z)\Psi_-(w) = (1 - \frac{w}{z})^{-2}\Psi_-(w)\Psi_+(z).$$

On the other hand the Goldstone vector $\xi_+ \in \mathcal{H}_j[k]$ is proportional to $E(0)^{m+2k}\xi_{-m-k}$ and the Goldstone vector $\xi_- \in \mathcal{H}_j[-k]$ is proportional to $E(0)^m\xi_{-m-k}$. Let $p = p_\pm = m + k \pm k$. Thus $\xi_\pm$ is proportional to $E(0)^{p_\pm}\xi_{-m-k}$. Now $\Psi(z) \equiv \Psi^1(z) = \sum E(n)z^{-n-1}$. Hence $\xi_\pm$ is proportional to the coefficient of $z_1^{-1} \dots z_p^{-1}$ in $\Psi(z_1) \dots \Psi(z_p)\xi_{-k-m}$. On the other hand if $C(z) = \sum c_n z^n$, then

$$\begin{aligned}\Psi(z_1) \dots \Psi(z_p)\xi_{-k-m} &= \prod_{i<j}(1 - \frac{z_i}{z_j})^2 z_1^{2(p-1)} z_2^{2(p-2)} \cdots z_p^2 (z_1 \cdots z_p)^{-2k-2p}\Psi_-(z_1)\cdots\Psi_-(z_p)\xi_{\pm k} \\ &= \prod_{i<j}(z_1 z_2 \cdots z_p)^{-2k-2m+p} C(z_1) \dots C(z_p) \prod_{i<j}(z_i - z_j)(z_i^{-1} - z_j^{-1})\xi_{\pm k}.\end{aligned}$$

Now we already noted that

$$C(z_1) \dots C(z_p) = \sum_{g_{p+1}=0} \chi_g(z) \cdot \det c_{g_i - i + j}.$$

Thus we are looking for the constant coefficient in

$$(z_1 z_2 \cdots z_p)^{-2k-2m+p} C(z_1) \dots C(z_p) \prod_{i<j}(z_i - z_j)(z_i^{-1} - z_j^{-1})\xi_{\pm k}.$$

By the combinatorics of the Weyl integration formula, the constant coefficient in

$$\chi_f(z^{-1})\chi_g(z)\prod_{i<j}(z_i - z_j)(z_i^{-1} - z_j^{-1})$$

equals $p!\delta_{f,g}$ (the usual orthogonality relations). Thus ξ_- is proportional to $X_f\xi_{-k}$ with $f_1 = 2k + m, \dots, f_m = 2k + m$ and $f_i = 0$ for $i > m$; and ξ_+ is proportional to $X_{f'}\xi_k$ with $f_1' = m, \dots, f_{2k+m}' = m$ and $f_i' = 0$ for $i > 2k + m$. Thus f' is the transposed Young diagram.

Following Segal, we now check this directly without using any unproved combinatorial results on symmetric functions. We have

$$\Delta(z^{-1}) = \det z_i^{-(p-j)}$$

and similarly

$$\Delta(z) = \sum_{\sigma \in S_p} \varepsilon(\sigma) z_{\sigma(1)}^{p-1} z_{\sigma(2)}^{p-2} \cdots z_{\sigma(p-1)}.$$

We are looking for the constant term in

$$C(z_1) \cdots C(z_p)(z_1 \cdots z_p)^{-2k-2m+p}\Delta(z^{-1}) \sum_{\sigma \in S_p} \varepsilon(\sigma)\sigma[z_1^{p-1} \cdots z_{p-1}].$$

Since $\Delta(z^{-1})$ satisfies $\sigma\Delta(z^{-1}) = \varepsilon(\sigma)\Delta(z^{-1})$ and the other terms are invariant under permutation, this is the same as the constant term in

$$\begin{aligned}p!C(z_1) \cdots C(z_p)&(z_1 \cdots z_p)^{-2k-2m+p} \det(z_i^{-(p-j)})\, z_1^{p-1} \cdots z_{p-1} \\ &= p!C(z_1) \cdots C(z_p) \det z_i^{j-2k-2m+p-i}\end{aligned}$$

But, if the determinant is written as an alternating sum over the symmetric group S_p, this is clearly proportional to $\det c_{2k+2m-p+i-j}$, as required.

Remark. The proof above is formally similar to the proof of the Weyl character formula for $U(N)$ (or $SU(N)$). Wallach made this link explicit by noticing that the vertex operator formalism allows a map to be defined from central functions on $U(n)$ into the oscillator representations. This map intertwines the action of Virasoro operators with the natural action of the operators ℓ_n $(n > 0)$ on central functions given by by $\ell_n = \mathcal{D}_n + \alpha \mathrm{Tr}(g^n)$ where $\mathcal{D}_n f(g) = d/dt f(g \exp -tg^n)|_{t=0}$.

We will now exhibit explicitly primary fields by compressing the primary fields $\Psi_k(z)$ on $\mathcal{H}_j$ between Goldstone vectors. This will reduce to the computation of the expressions of the form $L_1^{|f|} X_f \xi_p$. Note that since X_f has total degree f, we have

$$\frac{1}{|f|!} L_1^{|f|} X_f \xi_p = \alpha_f(p) \xi_p,$$

where $\alpha_f(p)$ is a constant, which we now determine through a product formula.

So we consider now the action of L_1 in the oscillator representation with $b_0 = 2\mu I$, so that$H(0) = \mu I$. This can be identified with the action on $\mathbb{C}[X]$ with $b_{-n} = \sqrt{2} X_n$ and $b_n = \sqrt{s} 2n \partial_{X_n}$ for $n > 0$. The operator L_1 is given by the formula

$$\begin{aligned} L_1 &= \frac{1}{4} \sum_{p+q=1} b_p b_q \\ &= \frac{1}{2} b_0 b_1 + \frac{1}{2} \sum_{p \geq 1} b_{-p} b_{p+1} \\ &= \mu \partial_{X_1} + \sum_{n \geq 2} n X_{n-1} \partial_{X_n}. \end{aligned}$$

This operator can thus be described in terms of two derivations D_1 and D_2 on the ring $\mathbb{C}[X]$, namely $D_1 X_n = n X_{n-1}$ $(n \geq 2)$, $D_1 X_1 = 0$ and $D_2 = \partial_{X_1}$. Now if $f(z) = \exp \sum_{n>0} X_n t^n / n = \sum c_n t^n$, then $D_1 f = (\sum_{n>0} X_n t^{n+1}) f = t^2 f'(t)$. It follows that $D_1 c_n = (n-1) c_{n-1}$. Similarly $D_2 f = tf$, so that $D_2 c_n = c_{n-1}$. Let $D = D_1 + \mu D_2$, also a derivation. Since X_f has degree $|f|$ and D decreases the degree by 1, it follows that $D^{|f|} X_f$ is a scalar.

Lemma F. $(|f|!)^{-1} D^{|f|} \det c_{f_i - i + j} = \det \binom{\mu + f_i - i + j - 1}{f_i - i + j}$.

Proof. Consider a term in the determinant $c_{m_1} \cdots c_{m_n}$. The term c_{m_i} has degree m_i so the product has degree $\sum m_i = |f|$. Since $D c_k = (k - 1 + \mu) c_{k-1}$, we have

$$D^k c_k = \mu(\mu+1) \cdots (\mu + k - 1) = k! \cdot \binom{\mu - 1 + k}{k}.$$

Hence by the Leibniz rule,

$$\begin{aligned} D^{|f|} c_{m_1} \cdots c_{m_n} &= \frac{|f|!}{m_1! \cdots m_n!} D^{m_1}(c_{m_1}) D^{m_2}(c_{m_2}) \cdots D^{m_n}(c_{m_n}) \\ &= |f|! \prod_{i=1}^{n} \binom{\mu - 1 + m_i}{m_i}. \end{aligned}$$

So applying $(|f|!)^{-1} D^{|f|}$ to the determinant has the effect of replacing each term c_k by $\binom{\mu - 1 + k}{k}$, as claimed.

Corollary. *If p is a half–integer, then $(|f|!)^{-1} L_1^{|f|} (X_f \xi_p) = \alpha_p(f) \xi_p$ where $\alpha_p(f) = \det \binom{2p + f_i - i + j - 1}{f_i - i + j}$.*

We now calculate the above determinant for the signature $f_i = N$ $(i = 1, m)$, $f_i = 0$ $(i > m)$.

Lemma G. *If $N \geq m$ then,*

$$\det \binom{\lambda + N - 1 + j - i}{N + j - i}_{i,j=1,m} = \frac{\prod_{j=1}^{N} \prod_{i=1}^{m} (\lambda - i + j)}{\prod_{j=1}^{N} \prod_{i=1}^{m} (N + i - j)}.$$

First proof (Weyl's character and dimension formulas). Recall that, if $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$ is a signature, then the character of the corresponding irreducible representation π_f of the unitary group $U(n)$ is given by

$$\chi_f(g) = \operatorname{Tr} \pi_f(g) = \frac{\det z_j^{f_i+n-i}}{\det z_j^{n-i}},$$

where $z_1, \ldots, z_n$ are the eigenvalues of g. The signature defines a Young diagram; the hook length of any node in the Young diagram is defined to be one plus the sum of the number of nodes in the same column or row after that node. If we place $n - i + j$ in at the jth node of the ith row, then the dimension d_f of π_f is the product of these numbers divided by the product of the hooklengths. It is equivalently given by the usual Weyl dimension formula

$$d_g = \frac{\prod_{j<i} (f_i - f_j - i + j)}{(n-1)!!},$$

although this formula is not manifestly written as a polynomial in n. If $f_1 = r$ and $f_i = 0$ for $i > n$, then χ_r is the character of the symmetric power $S^r \mathbb{C}^n$, which has dimension $\binom{n+r-1}{r}$. On the other hand χ_f can be expressed as a determinant in the χ_r's:

$$\chi_f = \det \chi_{f_i+j-i},$$

where we set $\chi_r = 0$ for $r < 0$. In particular evaluating the character at 1, we get a formula expressing a deteminant of binomial coefficients as a product of linear factors in n. Taking a Young diagram with m rows and $K + m$ boxes in each row, we have $f_i = K + m$ $(1 \leq i \leq m)$ and $f_i = 0$ for $i > m$. Hence

$$\det \binom{n - 1 + K + m + j - i}{K + m + j - i}_{i,j=1,\ldots,m} = \frac{\prod_{j=1}^{K+m} \prod_{i=1}^{m} (n - i + j)}{\prod_{j=1}^{K+m} \prod_{i=1}^{m} (K + m + i - j)}.$$

The stated determinantal formula follows by setting $\lambda = n$ and $N = K + m$.

Remark. Although we shall not need this result, the dimension formula implies more generally that $\det \binom{\mu+f_i-i+j-1}{f_i-i+j}$ is the product of factors $\mu + j - i$, when there is a box in the ith row and jth column of the Young diagram, divided by the product of hooklengths of the diagram. In other words if $\mu = -p$, with $p \geq 1$ an integer, then $(|f|!)^{-1} D^{|f|} X_f$ equals the dimension of the representation π_f of $U(p)$. It is therefore stictly positive when $f_{p+1} = 0$ and vanishes if $f_{p+1} \neq 0$. Since $L_{-1}\xi_0 = 0$, all these expressions vanish for $p = 0$.

Second proof. We have to show that

$$\det \binom{a + N + i - j}{N + i - j}_{i,j=1,m} = \prod_{p=1}^{N} \prod_{q=1}^{m} \frac{a + 1 + p - q}{N - p + q}.$$

We subtract a multiple of the second column from the first column to make the last entry of the first column 0; then we subtract a multiple of the third column from the second column to make the last entry of the second column 0, and so on. After these modification the (i, j) entry for $i, j < n$ is modified from $\binom{N+a+i-j}{a}$ to $\binom{N-1+a+i-j}{a}(m - i)/(N + m - j)$. Thus writing $F_m(a, N) = \det \binom{a+N+i-j}{N+i-j}_{i,j=1,m}$, we have

$$F_m(a, N) = \frac{\binom{a+N}{N}}{\binom{N+m}{m-1}} \cdot F_{m-1}(a - 1, N - 1).$$

This may be rewritten as

$$F_m(a,N)=\frac{(a+1)\cdots(a+N)}{m(m+1)\cdots(m+N-1)}\cdot F_{m-1}(a-1,N).$$

Thus

$$F_m(a,N)=(m-1)!!\frac{\prod_{i=1}^m\prod_{j=1}^N a+1+j-i}{\prod_{i=1}^m\prod_{j=1}^N(N+i-j)}.$$

Corollary. *If $f_i=N$ for $1\le i\le m$ and $f_i=0$ for $i>m$, then $(|f|!)^{-1}L_1^{|f|}(X_f\xi_p)=\alpha_f(p)\xi_p$ where*

$$\alpha_p(f)=\frac{\prod_{j=1}^N\prod_{i=1}^m(p-i+j)}{\prod_{j=1}^N\prod_{i=1}^m(N+i-j)}.$$

Lemma H. *For k an integer, $\Psi_k(z)\xi_0=e^{zL_{-1}}\xi_k$.*

Proof. Let $f(z)=\Psi_k(z)\xi_0$. We have $[L_{-1},\Psi_k(z)]=d\Psi_k/dz$ and $L_{-1}\xi_0=0$. Hence $df(z)/dz=L_{-1}f(z)$. Since $f(0)=\xi_k$, the result follows.

End of proof. We shall just use the fact that if f is a signature as above with $f_{p+1}=0$ with $p\ge1$, then

$$(L_1^{|f|}X_f\xi_p,\xi_p)\ne0.$$

We fix an integer $m\ge1$ and a half integer $k\ge0$. Since we can take adjoints, it will suffice to show that if i is a non–negative half integer with $i-k$ an integer and $|m-k|\le i\le m+k$, then there are are Goldstone vectors ξ_1,ξ_2 in $\mathcal{H}_j$ with ξ_1 having energy k^2 or i^2 and ξ_2 energy i^2 or k^2 such that $(\Psi_m(z)\xi_1,\xi_2)\ne0$. In this case if P_1 is the projection onto the $\mathfrak{vir}$–module generated by ξ_i, then $P_2\Psi_m(z)P_1$ or its adjoint is the required primary field of type m^2 between $L(1,k^2)$ and $L(1,i^2)$.

Assume first that k is an integer and set $s=|k-m|$. Take ξ_k in H_0 or $H_{1/2}$ and suppose i satisfies $|k-m|\le i\le k+m$ and $i-k\in\mathbb{Z}$. We assume further that $m\ge k$ and $i\ge m$ and set

$$A(z)=(\Psi_m(z)\xi_{-i},X_f\xi_{-i+m}),$$

where $X_f\xi_{-i+m}$ is a Goldstone vector of energy k^2. Thus $f_r=k+i-m$ for $r=1,\dots,k+m-i$ and $f_r=0$ for $r>k+m-i$. We must show that $A(z)\ne0$. But

$$\begin{aligned}A(z)&=(\Psi_m(z)V^i\xi_0,X_fV^{i-m}\xi_0)\\&=z^{-2mi}(\Psi_m(z)\xi_0,X_f\xi_m)\\&=z^{-2mi}(e^{zL_{-1}}\xi_m,X_f\xi_m)\\&=z^{-2mi}(\xi_m,e^{zL_1}X_f\xi_m).\end{aligned}$$

This is non–zero because $f_{m+1}=0$ since $m+1>k+m-i$.

Now suppose that $m\ge k$ and $i\le m$. In this case we consider

$$B(z)=(\Psi_m(z)\xi_{-k},X_f\xi_{m-k}),$$

where $X_f\xi_{m-k}$ is a Goldstone vector of energy i^2. Thus $f_r=m-k+i$ for $r=1,\dots,i+k-m$ and $f_r=0$ for $r>i+k-m$. Then

$$\begin{aligned}B(z)&=(\Psi_m(z)V^k\xi_0,X_fV^{-m+k}\xi_0)\\&=z^{-2km}(\Psi_m(z)\xi_0,X_f\xi_m)\\&=z^{-2km}(e^{zL_{-1}}\xi_m,X_f\xi_m)\\&=z^{-2km}(\xi_m,e^{zL_1}X_f\xi_m),\end{aligned}$$

which again is non–vanishing since $m \geq i+k-m$ and hence $f_{m+1}=0$.

Thus if $k \leq m$ we have shown that there is a non–zero primary field of type m^2 from $L(1,k^2)$ to $L(1,i^2)$ if $V_i \leq V_m \otimes V_k$. taking adjoints it follows that there is a non–zero primary field of type m^2 from $L(1,i^2)$ to $L(1,k^2)$ if $i \geq m$ and $k \leq m$ and $V_k \leq V_m \otimes V_i$. So now assume that $k \geq m$ and $i \geq m$ with $V_i \leq V_m \otimes V_k$. Since may take adjoints, we many assume that $i \geq k$. But then the computation with $A(z)$ above shows that the corresponding primary field exists.

Now suppose that $k \in 1/2+\mathbb{Z}$. Let W be a shift operator such that $W\xi_0 = \xi_{1/2}$. Let $k = p+\frac{1}{2}$ and $i = q+\frac{1}{2}$. If $k<m$ and $i>m$ or $k>m$ and $i>m$, then as before we take

$$A(z) = (\Psi_m(z)\xi_{-i}, X_f \xi_{-i+m}),$$

with the same choice of f as before. Then we have

$$\begin{aligned}
A(z) &= (\Psi_m(z)V^q W\xi_0, X_f V^{q-m} W\xi_0)\\
&= z^{-2im}(\Psi_m(z)\xi_0, X_f\xi_m)\\
&= z^{-2im}(e^{zL_{-1}}\xi_m, X_f\xi_m)\\
&= z^{-2im}(\xi_m, e^{zL_1}X_f\xi_m),
\end{aligned}$$

which as before is non–zero. Thus the primary fields exist in this case. If $i<m$ and $k<m$ we take

$$B(z) = (\Psi_m(z)\xi_{-k}, X_f\xi_{m-k}),$$

with the same choice of f as before. In this case

$$\begin{aligned}
B(z) &= (\Psi_m(z)V^p W\xi_0, X_f V^{p-m} W\xi_0)\\
&= z^{-2km}(\Psi_m(z)\xi_0, X_f\xi_m)\\
&= z^{-2km}(e^{zL_{-1}}\xi_m, X_f\xi_m)\\
&= z^{-2km}(\xi_m, e^{zL_1}X_f\xi_m),
\end{aligned}$$

which as before is non–zero. It follows that in all cases there is a non–zero primary field of type m^2 between $L(1,k^2)$ and $L(1,i^2)$ if $V_i \leq V_m \otimes V_k$.

9. Proof of the character formula for $L(1,j^2)$ using the Jantzen filtration. Let $A(x)=\sum_{i\geq 0} A_i x^i$ be an analytic family of non–negative self–adjoint matrices defined for x real and small on $V=\mathbb{R}^n$ or $\mathbb{C}^n$. We define a filtration $V^{(i)}$ ($i \geq 0$) by $V^{(0)} = V$ and for $m \geq 1$

$$V^{(m)} = \bigcap_{i=0}^{m-1} \ker A_i.$$

In the examples $A(x)$ will be invertible for $x \neq 0$ in a neighbourhood of 0. It follows that $V^{(m)}=(0)$ for m sufficiently large. We call $(V^{(m)})$ the *Jantzen filtration* associated with $A(x)$.

Lemma. *The order of* 0 *as a root of* $\det A(x)$ *equals* $k=\sum_{i\geq 1}\dim V^{(i)}$.

Proof. We set $U_0 = V^{(0)} = V$, $U_1 = V^{(n_1)}$ the first $k>0$ with $V^{(k)} \neq V^{(0)}$, $U_2 = V^{(n_2)}$ the first $k>n_1$ with $V^{(k)} \neq V^{(n_1)}$, and so on. We set $n_0=0$. Let $v_1,\dots,v_{m_1}$ be an orthonormal basis of $U_0 \ominus U_1$, $v_{m_1+1},\dots,v_{m_2}$ an orthonormal basis of $U_1 \ominus U_2$ and so on. Thus $(A(x)v_i,v_j) = x^{n_s}(A_{n_s}v_i,v_j) +$ higher powers of x for $i>m_s$. Moreover the first term vanishes if $j>m_{s+1}$. On the other hand the matrix $(A_{n_s}v_i,v_j)_{m_s<i,j\leq m_{s+1}}$ is invertible. Thus we can divide rows $m_s+1,\dots m_{s+1}$ by x^{n_s} and then set $x=0$. The resulting matrix is block triangular with invertible blocks on the diagonal, so is itself invertible. Hence the order of 0 as a root of $\det A(x)$ equals

$$k=\sum n_s\cdot(\dim U_{s-1}-\dim U_s) = \sum_{i\geq 1}\dim V_i,$$

as claimed.

Note that multiplying a bilinear form by (or matrix $A(x)$) by a factor $c_0 + c_1 x + c_2 x^2 + \cdots$ with $c_0 \neq 0$ does not change the Jantzen filtration. In this case we will say that the forms are *equivalent*.

We shall need an additional simple functoriality property of the Janzten filtration:

Lemma. *Let V, W be finite–dimensional inner product spaces and for $x \in (-\varepsilon, \varepsilon)$. let $A(x)$, $B(x)$ be self–adjoint non–negative matrices depending analytically on x. Let $X(x) \in \mathrm{Hom}(V,W)$ and $Y(x) \in \mathrm{Hom}(W,V)$ be analytic functions of x such that $(X(x)A(x)v,w) = (B(x)Y(x)v,w)$ for all $v \in V, w \in W$. Then $X(0)$ and $Y(0)^*$ induce maps of the corresponding Jantzen filtrations which are dual to each other on $V^{(i)}/V^{(i+1)}$ abd $W^{(i)}/W^{(i+1)}$.*

Proof. The condition implies that $X(x)A(x) = B(x)Y(x)$. If $A(x) = \sum A_n x^n$, $B(x) = \sum B_n x^n$, $X(x) = \sum X_n x^n$ and $Y(x) = \sum Y_n x^n$, then

$$\sum_{i+j=n} X_i B_j = \sum_{i+j=n} A_i Y_j. \qquad (*)$$

Thus $X_0 A_0 = B_0 Y_0$ on V. The equation $(*)$ implies that $X_0 A_i = B_i Y_0$ on $V^{(i)} = \ker A_0 \cap \cdots \cap \ker A_{i-1}$. But then X_0 and Y_0^* induce dual maps between $V^{(i)}/V^{(i+1)}$ and $W^{(i)}/W^{(i+1)}$, as claimed.

Now consider a representation $M(1,j^2)$. The point $(c,h) = (1,j^2)$ lies on the curve $\varphi_{p,1}(c,h) = 0$ where $p = 2j+1$. The curve is parametrized by a variable t via $c(t) = 13 - 6t - 6t^{-1}$ and $h(t) = (j^2+j)t - j$, the point $(1,j^2)$ corresponding to the value $t = 1$. If $t \neq 0, \infty$, the Verma module $M_t = M(c(t),h(t))$ is defined. It is a direct sum of finite energy spaces $M_t(n)$ on which L_0 acts as multiplication by $n + h(t)$. The space M_t has a canonical invariant bilinear form $B_t(v,w)$ defined on it. Invariance implies that the subspaces $M_t(n)$ are orthogonal. If we take as basis of $M_t(n)$ elements $v_i = L_{-i_r} \cdots L_{-i_1} v_t$ with $i_1 \leq i_2 \leq \cdots$ and $\sum s \cdot i_s = n$, then all these spaces can be identified with the same space $V = \mathbb{R}^{\mathcal{P}(n)}$. Thus $A_{ij}^{(n)}(t) = B_t(v_i, v_j)$ is a symmetric matrix, the entries of which are polynomials in t and t^{-1} with real coefficients. Since the Shapovalov form is invariant under the Virasoro algebra, it follows from the second lemma above that of $M = M(1,j^2)$ then, for $t = 1 + x$ with x small, the Jantzen filtration defines a filtration of M by $\mathfrak{vir}$–submodules $M^{(i)}$. In particular we can calculate $\sum_{i \geq 1} \dim M^{(i)}(n)$. It is exactly the degree of $t = 1$ as a root of the Kac determinant at level n which is known explicitly.

We now use the Janzten filtration to compute the character of $M(1,j^2)$. From the coset construction, we know that $M(1,j^2)$ contains a Verma submodule $M(1,(j+1)^2)$; this is also immediate from the Kac determinant formula for $c = 1$:

$$\det{}_N(1,h) = \prod_{p,q;\, pq \leq N} (h - \frac{(p-q)^2}{4})^{\mathcal{P}(N-pq)}.$$

Continuing in this way, it has a decreasing chain of Verma submodules $M(1,(j+k)^2)$ with $k \geq 1$. If we fix $h = j^2$ and set $c = 1 + x$, then as a function of x the determinant is proportional to

$$\Psi_N(x) = \prod_{1 \leq r \leq N} (j^2 + \frac{r^2-1}{24} \cdot x) \cdot \prod_{1 \leq s \leq r \leq N} [(j^2 - \frac{(r-s)^2}{4})^2 + ((r^2+s^2-2)j^2 + \frac{1}{2}(rs+1)(r-s)^2) \cdot \frac{x}{24} + (r^2-1)(s^2-1)(\frac{x}{24})^2]. \qquad (1)$$

We take x as the parameter of the Jantzten filtration. Since the operators L_n depend polynomially on the parameters t and t^{-1}, and hence analytically on x, it is evident that when $h = j^2$, they preserve the filtration $M^{(i)}$ of $M = M(1,j^2)$. If $j \geq 0$, then from (1) we get

$$X_{j^2}(q) \equiv \sum_{i \geq 1} \mathrm{ch}\, M^{(i)} = \sum_{N \geq 0} a(N) q^{N+j^2}$$

where

$$a(N) = \sum_{1\le r\le N,} \mathcal{P}(N - r(r+2j)).$$

Thus we obtain

$$X_{j^2}(q) = \sum_{i\ge 1} \operatorname{ch} M^{(i)} = \varphi(q)\cdot\left(\sum_{r\ge 1} q^{(r+j)^2}\right). \qquad (2)$$

Let v_r be the singular vector in $M(1,j^2)$ of energy $(j+r)^2$. Let $n_r > 0$ be maximal subject to $v_r \in M^{(n_r)}$. Thus the Verma module generated by v_r lies in $M^{(n_r)}$ and $\beta(w_1,w_2) = (w_1,w_2)_x/x^{n_r}|_{x=0}$ restricts to a multiple of the Shapovalov form. In particular n_r is the order of the zero of $(v_r,v_r)_x$. (Note that for $x>0$ the Shapovalov form is positive definite, so that $(v_r,v_r)_x > 0$ for $x>0$.) Since the Shapovalov form vanishes on any submodule, we have $\beta(v_{r+1},v_{r+1}) = 0$ and hence $n_r < n_{r+1}$. Moreover $M(1,(j+r)^2) \subseteq M^{(i)}$ for $n_{r-1} < i \le n_r$. Hence

$$X_{j^2}(q) \ge \sum_{N\ge 1} \operatorname{ch} M^{(i)} \ge \sum_{r\ge 1}(n_r - n_{r-1})\operatorname{ch} M(1,(j+r)^2) = \varphi(q)\cdot\sum_{r\ge 1}(n_r-n_{r-1})q^{(j+r)^2}, \qquad (3)$$

where the inequality is between coefficients of q^k and we set $n_0 = 0$. Comparing (3) with (1) and (2), it follows that $n_r = r$ for $r \ge 1$ and that equality holds in (3). Hence $M^{(1)} = M(1,(j+1)^2)$ and

$$\operatorname{ch} L(1,j^2) = (q^{j^2} - q^{(j+1)^2})\varphi(q),$$

as required.

10. Proof of the character formula for the discrete series $0<c<1$ using the Jantzen filtration. The proof of the character formula for the discrete series

$$c = 1 - \frac{6}{m(m+1)},\ h = h_{r,s} = \frac{(r(m+1)-sm)^2 - 1}{4m(m+1)},\ (1\le s\le r\le m-1)$$

is proved similarly using $x = h - h_{r,s}$ as a parameter. Let

$$\varphi_{r,r}(c,h) = h + \frac{1}{24}(r^2-1)(c-1)$$

and for $r,s\ge 1$ with $r\ne s$,

$$\varphi_{r,s}(c,h) = \left(h - \frac{(r-s)^2}{4}\right)^2 + \frac{h}{24}(r^2+s^2-2)(c-1) + \frac{1}{24^2}(r^2-1)(s^2-1)(c-1)^2 + \frac{1}{48}(c-1)(r-s)^2(rs+1).$$

We shall need the following result, implicit in **[8]**. We give two proofs, the second of which is an expanded and slightly corrected version of a result from **[13]**.

Proposition (Fuchs algorithm). *If $\varphi_{r,s}(c,h) = 0$, then $M(c,h)$ has a non–zero singular vector at energy level $h+rs$.*

Proof 1. A singular vector at energy level $h+rs$ has the form Pv_h where

$$P_d = \sum_{k=0}^{rs} q_k L_{-1}^k.$$

Each term q_k is a sum of monomials $L_{-k}^{n_k}\cdots L_{-2}^{n_2}$ where $2n_2 + 3n_3 + \cdots = d-k$. By the uniqueness result of Fuchs, we may assume $q_d = 1$, where $d = rs$. Set $n_1 = d-k$ and order these momomials lexigraphically on $(n_2, n_3, \ldots)$. Thus the first term is L_{-1}^d.

Consider a term $\cdots L_{-p-1}^{n_{p+1}} L_{-p}^{n_p} L_{-1}^{d-n_1} v$ with $n_p > 0$ and suppose that we have already determined the coefficients of all previous monomials in the lexicographic order to be polynomials in c and h. Let a be the coefficient of $Bv = \cdots L_{-p-1}^{n_{p+1}} L_{-p}^{n_p} L_{-1}^{d-n_1} v$ in the singular vector w. Let $k = d - n_1$. Then since fot a singular vector we would require $L_{p-1} w = 0$, in this case we just require that the coefficient of

$$\cdots L_{-p-1}^{n_{p+1}} L_{-p}^{n_p - 1} L_{-1}^{k+1} v \tag{$*$}$$

in $L_{p-1} w$ must be zero. We claim that this coefficient equals $n_p(2p-1)a$ plus a sum of lower order coefficients times polynomials in c and h. In particular a is again a polynomial in c and h.

If A is an element of the enveloping algebra of the Virasoro algebra we shall write $Av \sim 0$ if Av is a combination of terms lower than Bv in the lexicographic order, with coefficients polynomials in c and h. Now we have

$$L_{p-1} w = \sum_{j \geq 0} L_{p-1} q_j L_{-1}^j v = \sum_{k \geq 0} [L_{p-1}, q_j] L_{-1}^j v + q_j L_{p-1} L_{-1}^j v.$$

We look for terms ending with $L_{-p}^{n-1} L_{-1}^{k+1}$ in this expression. We first note that it follows by induction on $k \geq 1$ that if $j \geq 1$ then $L_j L_{-1}^k v$ lies in $\mathrm{lin}\,\{A L_{-1}^i v : i < k,\ A \in \mathcal{U}_2\}$. Indeed

$$L_j L_{-1} L_{-1}^{k-1} v = (j+1) L_{j-1} L_{-1}^{k-1} v + L_{-1} L_j L_{-1}^{k-1} v,$$

which has the same form since $L_{-1} \mathcal{U}_2 = \mathcal{U}_2 L_1$ and $L_{-1}^{k-1} v$ is an eigenvector of L_0. If $k \neq c$, then the induction hypothesis forces $q_k \sim D$ with $D \in \mathcal{U}_p$. So either $k > c$, in which case any monomial q_k could have a non–zero contribution from a monomial $L_{-s}^{m_s} \cdots L_{-p}^{m_p}$ with $m_p > n_p$. Clearly taking the Lie bracket with L_{p-1} can increase the exponent of L_{-1} by at most one while diminishing the exponent of L_{-p} by at most one. So monomials ending with $L_{-p}^{n-1} L_{-1}^{c+1}$ have coefficients that are polynomials in c and h. If $k < c$, there is no way to increase the power of L_{-1}^k to L_{-1}^{c+1} by taking the Lie bracket with L_{p-1}. For $k = c$ and the terms $[L_{p-1}, A L_{-p}^n]$ with A a monomial in $\mathcal{U}_{p+1}$, we have

$$[L_{p-1}, A L_{-p}^n] = [L_{p-1}, A] L_{-p}^n + A [L_{p-1}, L_{-p}^{n_p}].$$

The first term lies in $\mathcal{U}_2$ while for the second

$$[L_{p-1}, L_{-p}^n] v = (2p-1) \sum_{a+b=n-1} L_{-p}^a L_{-1} L_{-p}^b v = (2p-1) n L_{-p}^{n-1} L_{-1} + B,$$

where $B \in \mathcal{U}_2$. There could be several terms in q_c with $n_p = n$; but in this particular case on bracketing with L_{p-1} and taking the term ending with $L_{-p}^{n-1} L_{-1}^{c+1}$ we obtain $A L_{-p}^{n-1} L_{-1}^{c+1}$. It follows that all these terms are linearly independent. But then there can be no cancellation and the coefficient of $L_{-r}^{n_r} \cdots L_{-p}^{n_p - 1} L_{-1}^{c+1} v$ must be $n_p(2p-1)a$ plus a sum of lower order coefficients times polynomials in c and h, as claimed.

Now given this vector $w = \sum a_\alpha(c,h) L_\alpha L^{d-|\alpha|}$ where the coefficients $a_\alpha(c,h)$ are polynomials in c and h, the condition $L_1 w = 0 = L_2 w$ gives a series of polynomials $b_i(c,h)$ which must vanish for w to be a singular vector. By the coset construction, we know that $L(c_m, h_{r,s}(q))$ has a singular vector at energy $h_{r,s}(m) + rs$ if $1 \leq s \leq r \leq m-1$. Thus $b_i(c_m, h_{r,s}(m)) = 0$. Thus $b_i(c,h)$ vanishes at a point of accumulation on the real curve $\varphi_{r,s}(c,h) = 0$. It follows that $b_i(c,h) = 0$ whenever $\varphi_{r,s}(c,h) = 0$. Thus w defines a non–zero singular vector if $\varphi_{r,s}(c,h) = 0$, as required.

Proof 2. A singular vector at energy level $h + rs$ has the form $P v_h$ where

$$P_d = \sum_{k=0}^{rs} q_k L_{-1}^k.$$

Each term q_k is a sum of monomials $L_{-k}^{n_k} \cdots L_{-2}^{n_2}$ where $2n_2 + 3n_3 + \cdots = d - k$. By the uniqueness result of Fuchs, we may assume $q_d = 1$, where $d = rs$. Set $n_1 = d - k$ and order these momomials lexigraphically on $(n_1, n_2, n_3, \ldots)$. Thus the first term is L_{-1}^d.

Consider a term $\cdots L_{-p-1}^{n_{p+1}} L_{-p}^{n_p} L_{-1}^{d-n_1} v$ with $n_p > 0$ and suppose that we have already determined the coefficients of all previous monomials in the lexicographic order to be polynomials in c and h. Let a be the coefficient of $Bv = \cdots L_{-p-1}^{n_{p+1}} L_{-p}^{n_p} L_{-1}^{d-n_1} v$ in the singular vector w. Let $k = d - n_1$. Then since fot a singular vector we would require $L_{p-1} w = 0$, in this case we just require that the coefficient of

$$\cdots L_{-p-1}^{n_{p+1}} L_{-p}^{n_p - 1} L_{-1}^{k+1} v \tag{$*$}$$

in $L_{p-1} w$ must be zero. We claim that this coefficient equals $n_p(2p-1)a$ plus a sum of lower order coefficients times polynomials in c and h. In particular a is again a polynomial in c and h.

If A is an element of the enveloping algebra of the Virasoro algebra we shall write $Av \sim 0$ if Av is a combination of terms lower than Bv in the lexicographic order, with coefficients polynomials in c and h.

We have

$$L_{p-1} w = \sum_{k=0}^{e} L_{p-1} q_k L_{-1}^k v = \sum_{k=0}^{e} [L_{p-1}, q_k] L_{-1}^k v + q_k L_{p-1} L_{-1}^k v.$$

We look for terms ending with $L_{-p}^{n-1} L_{-1}^{e+1}$ in this expression. We first note that it follows by induction on k that if $j \geq 1$ then $L_j L_{-1}^k v$ lies in $\mathrm{lin}\,\{A L_{-1}^i v : i < k,\ A \in \mathcal{U}_2\}$. Indeed

$$L_j L_{-1} L_{-1}^{k-1} v = (j+1) L_{j-1} L_{-1}^{k-1} v + L_{-1} L_j L_{-1}^{k-1} v,$$

which has the same form since $L_{-1} \mathcal{U}_2 = \mathcal{U}_2 L_1$ and $L_{-1}^{k-1} v$ is an eigenvector of L_0.

By the inductive hypothesis, all terms with $k > e$ are have uniquely determined polynomial coefficients. A monomial with $k \leq e$ can be written $u = B \cdot L_{-p}^n A \cdot L_{-1}^k v$ with B a monomial in L_{-j}'s for $j > p$ and A a monomial in the L_{-j}'s with $2 \leq < p$. But then

$$L_{p-1} u = [L_{p-1}, B] \cdot L_{-p}^n A L_{-1}^k v + B \cdot (L_{p-1}, L_{-p}^n] \cdot A L_{-1}^k v) + B L_{-p}^n L_{p-1} A L_{-1}^k v. \tag{1}$$

Since $[L_{p-1}, B]$ lies in $\mathcal{U}_2$, the first term in this expression has no terms ending in L_{-1}^{e+1}. For the second note that

$$[L_{p-1}, L_{-p}^n] = (2p-1) \sum_{a+b=n-1} L_{-p}^a L_{-1} L_{-p}^b = (2p-1) n L_{-p}^{n-1} L_{-1} + D,$$

where $D \in \mathcal{U}_2$. Clearly, if $k < e$, then neither $BDL_{-1}^k v$ nor $BL_{-1} A L_{-1}^k v$ contain terms ending in $L_{-1}^{e+1} v$. Finally for the last term, note that if X is a monomial in L_{-j} for $1 \leq j \leq t$, then for $1 \leq s \leq t$ we have $L_s X v = \sum a_i X_i v$, where X_i has the same form as X with the exponent of L_{-1} increased by at most one. Thus no terms ending in $L_{-p}^{n_p - 1} L_{-1}^{e+1}$ arise this way if $k < e$.

When $k = e$, then by the inductive hypothesis, we only need to consider terms of the form $u = B \cdot L_{-p}^n L_{-1}^e v$ with either $n \geq n_p > 0$ or zero. If $n = 0$, then

$$L_{p-1} u = L_{p-1} B L_{-1}^e v = [L_{p-1}, B] L_{-1}^e v + B L_{p-1} L_{-1}^e v.$$

The first term contains no terms ending in $L_{-1}^{e+1} v$. Nor does the second, since if $kj0$, $L_j L_{-1}^e v$ lies in $\mathrm{lin}\,\{L_{-1}^i v : 0 \leq i < e\}$. If $n > n_p$, the none of the terms in (1) give rise to a monomial ending in $L_{-1}^{n_p - 1} L_{-1}^{e+1} v$.

Finally there could be several terms $BL_{-p}^n L_{-1}^e$ in q_e with $n = n_p$. However taking the term ending with $L_{-p}^{n-1} L_{-1}^{e+1} v$ in (1), we get $BL_{-p}^{n-1} L_{-1}^{e+1} v$. Thus all these terms are distinct. But then there can be no cancellation and and thus the coefficient of $L_{-r}^{n_r} \cdots L_{-p}^{n_p - 1} L_{-1}^{c+1} v$ must be $n_p(2p-1)a$ plus a sum of lower order coefficients times polynomials in c and h.

Now given this vector $w = \sum a_\alpha(c,h) L_\alpha L_{-1}^{d-|\alpha|}$ where the coefficients $a_\alpha(c,h)$ are polynomials in c and h, the condition $L_1 w = 0 = L_2 w$ gives a series of polynomials $b_i(c,h)$ which must vanish for w to be a singular vector. By the coset construction, we know that $L(c_m, h_{r,s}(q))$ has a singular vector at energy $h_{r,s}(m) + pq$ if $1 \leq s \leq r \leq m-1$. Thus $b_i(c_m, h_{r,s}(m)) = 0$. Thus $b_i(c,h)$ vanishes at a point of accumulation on the real curve $\varphi_{r,s}(c,h) = 0$. It follows that $b_i(c,h) = 0$ whenever $\varphi_{r,s}(c,h) = 0$. Thus w defines a non–zero singular vector if $\varphi_{r,s}(c,h) = 0$, as required.

Corollary. *If $r, s \geq 1$ and $c = 1 - 6/m(m+1)$, then $M(c, h_{r,s})$ has a singular vector of energy $h = h_{r,s} + rs$.*

Proposition. *Let $c = 1 - 6/m(m+1)$, $1 \leq s \leq r \leq m-1$, $r' = m - r$, $s' = m + 1 - q$. Then $M(c, h_{r,s})$ has singular vectors $a_1, b_1, a_2, b_2, \cdots$ where for $k \geq 1$, a_k has energy $\alpha_k = h_{r',k(m+1)+s'} = h_{r,s-k(m+1)}$ and b_k has energy $\beta_k = h_{r,s+k(m+1)} = h_{r',s'-k(m+1)}$. The energies of the vectors are strictly increasing in the above order. Moereover if A_k and B_k are the Verma modules generated by a_k and b_k, then $A_{k+1} + B_{k+1} \subseteq A_k \cap B_k$.*

Proof. Since

$$\alpha_{k+1} = \alpha_k + r'(k(m+1) + s'),\ \beta_{k+1} = \alpha_k + r(s - k(m+1))$$

and

$$\alpha_{k+1} = \beta_k + r(s + k(m+1)),\ \beta_{k+1} = \beta_k + r'(s' - k(m+1)),$$

existence follows by successive applications of the previous corollary. The uniqueness of singular vectors as a particular energy level in a Verma module implies the result on the set theoretic inclusion.

For fixed m and $1 \leq s \leq r \leq m-1$, take $x = h - h_{r,s,}(m)$. As in the previous section let $m_k > 0$ be the order of the zero of $(a_k, a_k)_x$ and n_k that of $(b_k, b_k)_x$. For this to make sense, we have to check that, if $\xi = a_k$ or b_k, then $(\xi, \xi)_x$ does not vanish identically. But m $(\xi, \xi)_x > 0$ for x sufficiently large, because the Kac determinant formula and the unitarity of Verma modules for $c \geq 1$ together imply that, at a fixed energy level, the Shapovalov form is positive definite for $c > 0$ and h sufficiently large. The same argument as used in the Section 9 shows that $m_k < m_{k+1}, n_{k+1}$ and $n_k < m_{k+1}, n_{k+1}$ since $A_{k+1} + B_{k+1} \subseteq A_k \cap B_k$.

Proposition. *$A_1 + B_1$ is the maximal submodule of $M = M(c, h_{r,s})$ and $A_{k+1} + B_{k+1} = A_k \cap B_k$ for $k \geq 1$. Moreover $m_k = n_k = 1$ and*

$$\sum_{i \geq 1} \mathrm{ch}\, M^{(i)} = \mathrm{ch}(A_1 + B_1) + \sum \mathrm{chA_{2k}} + \mathrm{ch}\, B_{2k} = \sum_{k \geq 0} \mathrm{ch}\, A_{2k+1} + \mathrm{ch}\, B_{2k+1}.$$

Proof. From the Kac determinant formula $h_{r,s} = h_{r_1,s_1}$ for $(r,s) = (r_1, s_1) + a(m, m+1)$ with $a \geq 0$ or $(r,s) = -(r_1, s_1) + b(m, m+1)$ with $b \geq 1$. Hence

$$\sum \mathrm{ch}\, M^{(i)} = \varphi(q) \cdot \sum_{a \in \mathbb{Z}} q^{(r+am)(s+a(m+1))}. \tag{1}$$

On the other hand if $M_k = \max(m_k, n_k)$ and $N_k = \min(m_k, n_k)$, then $N_k \leq M_k < N_{k+1} \leq M_{k+1}$. On the other hand

$$\sum \mathrm{ch}\, M^{(i)} \geq \sum_{k \geq 1} (N_{k+1} - M_k) \mathrm{ch}\, (A_k + B_k) \geq \sum_{k \geq 1} \mathrm{ch}\, (A_k + B_k). \tag{2}$$

Thus

$$\sum \mathrm{ch}\, M^{(i)} \geq \sum_{k \geq 1} \mathrm{ch}\, (A_k + B_k).$$

We wish to prove the same inequality with $A_k \cap B_k$ replacing $A_{k+1} + B_{k+1}$ for all k. Indeed suppose that R_k is chosen maximal such that $A_k \cap B_k \subseteq M^{(R_k)}$. Then $M_k + 1 \leq R_k < N_k$. Hence $R_k < R_{k+1}$ and so

$$\sum \mathrm{ch}\, M^{(i)} \geq \mathrm{ch}\, (A_1 + B_1) + \sum_{k \geq 1} (R_k - R_{k-1}) \mathrm{ch}\, A_k \cap B_k \geq \mathrm{ch}\, (A_1 + B_1) + \sum_{k \geq 1} \mathrm{ch}\, A_k \cap B_k.$$

Now we have a short exact sequence

$$0 \to A_k \cap B_k \to A_k \oplus B_k \to A_k + B_k \to 0.$$

Hence

$$\mathrm{ch}\, (A_1 + B_1) + \mathrm{ch}\, A_1 \cap B_1 = \mathrm{ch}\, A_1 + \mathrm{ch}\, B_1,$$

and for $k \geq 1$

$$\mathrm{ch}\, A_{2k} \cap B_{2k} + \mathrm{ch}\, A_{2k+1} \cap B_{2k+1} \geq \mathrm{ch}\, (A_{2k+1} + B_{2k+1}) + \mathrm{ch}\, A_{2k+1} \cap B_{2k+1} = \mathrm{ch}\, A_{2k+1} + \mathrm{ch}\, B_{2k+1}.$$

Hence

$$\sum \operatorname{ch} M^{(i)} \geq \operatorname{ch}(A_1 + B_1) + \sum_{k\geq 1} \operatorname{ch} A_k \cap B_k \geq \sum_{k\geq 0} \operatorname{ch} A_{2k+1} + \operatorname{ch} B_{2k+1}. \tag{3}$$

On the other hand the right hand side equals

$$\varphi(q) \cdot \sum_{a\in\mathbb{Z}} q^{(r+am)(s+a(m+1))},$$

so it coincides with the left hand side. So we have equality in (3), so that

$$A_{2k} \cap B_{2k} = A_{2k+1} + B_{2k+1}$$

for all k. Now instead of M we take $M' = A_1$ and set $A'_k = B_{k+1}$, $B'_k = A_{k+1}$ for $k \geq 1$. Repeating the argument above we get

$$\sum_{i\geq 1} \operatorname{ch}(M')^{(i)} \geq \sum_{k\geq 1} \operatorname{ch} A_k \cap B_k \geq \sum_{k\geq 1} \operatorname{ch} A_{2k} + \operatorname{ch} B_{2k}. \tag{4}$$

By (1), the left hand side of (4) equals

$$\varphi(q) \cdot \sum_{a\in\mathbb{Z}} q^{(r_1+am)(s_1+a(m+1))},$$

where $r_1 = r$ and $s_1 = -s$. But this equals the last term on the right hand side. Hence equality holds in (4), so that

$$A_{2k-1} \cap B_{2k-1} = A_{2k} + B_{2k}$$

for $k \geq 1$. Returning to equation (2), we have equality between the furthest terms and hence $N_{k+1} - M_k = 1$ for all k. Evidently $m_1 < m_2 < \cdots$ and $n_1 < n_2 < \cdots$. Now we have $M^{(1)} \supseteq A_1 + B_1$, so $m_1 = 1 = n_1$. Hence $M_1 = 1$. So $N_2 = \min(m_2, n_2) = 2$. Note that if $N_k < M_k$ then we would have an extra contribution of $\operatorname{ch} A_k$ or $\operatorname{ch} B_k$ on the right hand side of (2). Since equality holds, it follows that $N_k = M_k$, so that $n_k = m_k$. Since $m_1 = 1 = n_1$, we finally get $m_k = k = n_k$.

Corollary. $\operatorname{ch} L(c, h_{r,s}) = \varphi(q) \cdot \sum_{k\in\mathbb{Z}} (-1)^k q^{h_{r,s+k(m+1)}}$.

Proof. We have

$$\begin{aligned}
\operatorname{ch} L(c, h_{pq}) &= \operatorname{ch} M(c, h_{r,s})/(A_1 + B_1) \\
&= \operatorname{ch} M(c, h_{r,s}) - \operatorname{ch}(A_1 + B_1) \\
&= \operatorname{ch} M(c, h_{r,s}) + \sum_{k\geq 0} \operatorname{ch} A_{2k+1} + \operatorname{ch} B_{2k+1} - \sum_{k\geq 0} (\operatorname{ch} A_{2k+1} + \operatorname{ch} B_{2k+1}) \\
&= \varphi(q) \cdot \sum_{k\in\mathbb{Z}} (-1)^k q^{h_{r,s+k(m+1)}}.
\end{aligned}$$

APPENDIX A: Alternative proofs of Fubini–Veneziano relations.

First indirect proof. Recall that $\phi(n) : \mathcal{F} \to \mathcal{F}$ is a primary field for the system of operators (a_n), U and L_0 if

$$[a_n, \phi(k)] = m\phi(k+n), \ [L_0, \phi(n)] = -(n+\mu)\phi(n), \ U\phi(n)U^* = \phi(n+m),$$

for some $m \in \mathbb{Z}$ and $\mu \in \mathbb{R}$. We now fix k and consider

$$\psi_k(n) = [L_k, \phi(n-k)] + (n-k)\phi(n).$$

It is easy to check that ψ satisfies the same conditions as ϕ with the same choice of μ. So by uniqueness, $\psi_k(n) = c(k)\phi(n)$ for some constant $c(k)$:

$$[L_k, \phi(n-k)] + (n-k)\phi(n) = c(k)\phi(n). \tag{1}$$

The Jacobi relation, the relation $[L_a, [L_b, T]] - [L_b, [L_a, T]] = (a-b)[L_{a+b}, T]$ and the non–vanishing of $\phi(0)$ imply that

$$(a-b)c(a+b) = ac(a) - bc(b).$$

But this functional equation implies that c is an affine function $c(a) = \alpha a + \beta$. Indeed the c satisfying this functional equation form a vector space. So subtracting an affine function from c, we may assume that $c(0) = 0 = c(1)$ and must then show that $c \equiv 0$. Taking $a = -b$, we see that $c(a) = -c(-a)$ for all a. But for $a > 1$, $(a-1)c(a+1) = ac(a)$. Hence $c(a) = 0$ for all $a > 0$ and hence for all a. So from (1) we have

$$[L_k, \phi(n)] = (-n - \alpha k - \beta)\phi(n+k). \tag{2}$$

Now on the one hand we have $\Phi_m(z) = \sum \phi(n) z^{-n-\delta}$ while on the other, regardless of the labelling of the $\phi(n)$'s, we have

$$\Phi_m(z)\Omega|_{z=0} = \Omega_m.$$

It follows that $\delta = 0$ and $\phi(n)\Omega = 0$ for $n > 0$, with $\phi(0)\Omega = \Omega_m$. Since $L_0\Omega_m = \frac{m^2}{2}\Omega_m$, we must have $\beta = -m^2/2$. From equation (2) we have that

$$[L_k, \Phi_m(z)] = z^{k+1}\Phi_m'(z) + \Delta z^k(k+1)\Phi_m(z),$$

where $\Delta = 1 - \alpha$ and $\delta = \alpha - \beta - 1$. But $\delta = 0$. Hence $\alpha = 1 - m^2/2$ and $\Delta = m^2/2$. This proves the Fubini–Veneziano relations.

Second proof by direct verification. We start by noting the adjoint relation

$$(\Phi_m(z^{-1}))^* = z^{-m^2}\Phi_{-m}(z)$$

follows immediately because

$$\begin{aligned}
\Phi_m(z^{-1})^* &= \exp(\sum_{n<0} \frac{-z^n m a_n}{n}) \exp(\sum_{n>0} \frac{-m z^n a_n}{n}) z^{-ma_0} U^m \\
&= z^{-ma_0} U^m \exp(\sum_{n<0} \frac{-z^n m a_n}{n}) \exp(\sum_{n>0} \frac{-m z^n a_n}{n}) \\
&= z^{-m^2}\Phi_{-m}(z).
\end{aligned}$$

We shall need the following generalisation of Lemma B in Section 8.

Lemma. *Let A be a formal power series in z (or z^{-1}) with operator coefficients and let D be an operator such that $C = [A, B]$ is a multiple of the identity operator, where $B = [D, A]$. Then $[D, e^A] = (B + C/2)e^A$.*

Proof. We have

$$\begin{aligned}
[D, A^N] &= \sum_{p+q=N-1} A^p B A^q \\
&= \sum_{p+q=N-1} BA^{p+q} + pCA^{p+q-1} \\
&= NBA^{N-1} + \frac{N(N-1)}{2} CA^{N-2}.
\end{aligned}$$

Hence

$$[D, e^A] = (B + C/2)e^A.$$

Corollary. *(a)* $[L_k, E_+(z)] = (-\sum_{n>0} a_{n+k} z^{-n}) E_+(z)$,
and $[L_{-k}, E_-(z)] = (-\sum_{n>0} a_{-n-k} z^{-n}) E_-(z)$ *if* $k \geq 0$.
(b) $[L_{-k}, E_+(z)] = (-\sum_{n>0} a_{n-k} z^{-n} + \frac{m^2}{2}(k-1)z^k) E_+(z)$
and $[L_k, E_-(z)] = (-\sum_{n>0} a_{-n-k} z^{n} - \frac{m^2}{2}(k+1)z^{-k}) E_-(z)$ *for* $k > 0$.

Proof. These formulas are straightforward consequences of the lemma, setting $D = L_k$ and

$$A = \sum_{\pm n > 0} \frac{m}{n} a_n z^{-n}.$$

For example to prove the first formula in (b), we have

$$B = [D, A] = -\sum_{n>0} m a_{n-k} z^{-n} = -\sum_{i > -k} m a_i z^{-i+k}$$

so that

$$\begin{aligned} [A, B] &= -[\sum_{n>0} \frac{m}{n} a_n z^{-n}, \sum_{i>-k} m a_i z^{-i+k}] \\ &= -\sum_{n=1}^{k-1} [a_n, a_{-n}] \frac{m^2}{n} z^k \\ &= -m^2 (k-1) z^k. \end{aligned}$$

Proof of the Fubini–Veneziano relations. We first check the commutation relations with L_k for $k \neq 0$. For $k > 0$, we have $U L_{-k} U^* = L_{-k} + a_{-k}$, so that $U^m L_{-k} U^{-m} = L_{-k} + m a_{-k}$ and hence

$$[L_{-k}, U^m] = -m U^m a_{-k}.$$

Thus

$$[L_-k, \Phi_m(z)] = U^m z^{-m a_0} (B_- E_- E_+ + E_- B_+ E_+),$$

where

$$B_- = -m a_{-k} - \sum_{n>0} m a_{-n-k} z^{-n}, \; B_+ = -\sum_{n>0} m a_{n-k} z^{-n} - \frac{m}{2}(z^k - 1)/(z-1).$$

On the other hand

$$z^{-k+1} \Phi_m'(z) = U^m z^{-m a_0} z^{-k} (-m a_0) E_- E_+ + U^m z^{-m a_0} (C_- E_- E_+ + E_- C_+ E_+),$$

where

$$C_- = -\sum_{n>0} m a_{-n} z^{n-k}, \; C_+ = -\sum_{n>0} m a_n z^{-n-k}.$$

Thus

$$B_- = C_- - A$$

and

$$B_+ = C_+ + A - \frac{m^2}{2}(k-1) z^{-k} I$$

where

$$A = \sum_{i=0}^{k-1} m a_i z^{-i-k}.$$

On the other hand

$$\begin{aligned} [A, E_-(z)] &= \sum_{i=0}^{k-1} m [a_i, E_-(z)] z^{i-k} \\ &= \sum_{i=1}^{k-1} m^2 z^{-k} E_-(z) \\ &= m^2 z^{-k} (-1) k E_-(z). \end{aligned}$$

Hence

$$[L_{-k},\Phi_m(z)] - z^{-k+1}\Phi_m'(z) = [-\frac{m^2}{2}(k-1) + m^2(k-1)]\Phi_m(z)$$
$$= \frac{m^2}{2}(k-1)z^{-k}\Phi_m(z),$$

as required. The relation $[L_k,\Phi_m(z)]$ can be proved similarly or follows from this one by taking adjoints.

APPENDIX B: Explicit construction of singular vectors and asymptotic formulas of Feigin–Fuchs. In this appendix we give a slightly simplified account of the approach of **[3], [7]** to singular vectors in $M(c(t),h_{r,1}(t))$, where $c(t) = 13-6t-6t^{-1}$, $r = 2j+1$ for j a non–negative half integer and $h_{r,1}(t) \equiv h(t) = (j^2+j)t - j$. The case of interest in the text has $t = 1$, so that $c = 1$ and $h = j^2$. The treatment of **[8]**, however, uses the more general case, when the singular vector is a polynomial in t and a knowledge of the constant term and leading coefficient is required. A proof of their formulas in a more general setting was given in **[2]**, who commented that in the particular case the method of **[3]** for determining the singular vector could not be used to derive the result. We show on the contrary that the formulas are a trivial consequence of the algorithm of **[3]**, which provided an alternative approach to explicit formulas for the singular vectors of Benoit and St–Aubin **[4]**. Let E,F,H be a canonical basis of $\mathfrak{sl}_2$ satisfying $[E,F] = 2H$, $[H,E] = E$ and $[H,F] = -F$ and let $C = H^2+(EF+FE)/2$, the Casimir element. Let $V = V_j$ be the irreducible representation of $\mathfrak{sl}_2$ of spin j with (non–orthonormal) basis $v_{-j}, v_{-j+1},\cdots,v_j$ satisifying $Hv_k = kv_k$, $F^k v_j = v_{j-k}$ for $k > 0$ and $F^{2j+1}v_j = 0$. Let $W = \bigoplus_{m\in\mathbb{Z}} W(m)$ be a representation of the Virasoro algebra with $L_0 w = (h+m)w$ for $w \in W(m)$ and $\dim W(m) < \infty$ for all m. We make operators A on V and B on W act on $V\otimes W$ as $A\otimes I$ and $I\otimes B$ respectively. With this convention, we define

$$N = -F + \sum_{m\geq 0}(-tE)^m L_{-m-1},\ M = L_0 - H - tC,\ L = L_1 + E(t(H-1)+1),\ K = L_2 - tE^2(t(H-\frac{3}{2})+\frac{7}{4}).$$

Lemma A. *$[N,M] = aN$, $[N,L] = bEN + cM$ and $[N,K] = dE^2N + eEM + fL$ where $a = -1$, $b = -3t$, $c = -2$, $d = -5t^2$, $e = -4t$ and $f = -3$.*

Remark. More generally, if we define $\mathcal{L}_{-1} = N$, $\mathcal{L}_0 = M$ and

$$\mathcal{L}_k = L_k - (-E)^k t^{k-1}(t(H-\frac{k+1}{2}) + \frac{3k+1}{4})$$

for $k \geq 1$ (so that $\mathcal{L}_1 = L$ and $\mathcal{L}_2 = K$), then for $p \geq 0$

$$[\mathcal{L}_p,\mathcal{L}_{-1}] = \sum_{q\geq 0}(p+1+q)t^q E^q \mathcal{L}_{p-1-q}$$

and for $p,q \geq 0$

$$[\mathcal{L}_p,\mathcal{L}_q] = (p-q)\mathcal{L}_{p+q}.$$

Proof. This is a straightforward verification using the commutation relations.

Lemma B. *Let $\xi \in W$ satisfy $L_0\xi = h(t)\xi$. Then there is a unique vector $w = \sum_{k=0}^{2j+1} v_{-j+k}\otimes\xi_{-j+k}$, $\xi_{-j} = \xi$ with $L_0\xi_k = (h(t)+k)\xi_k$ and $Nw = v_j\otimes\eta$. In this case $\eta = P_j\xi_0$ with P_j a uniquely determined homogenenous polynomial of total degree $r = 2j+1$ in the universal enveloping algebra $\mathcal{U}_-$ of the Lie algebra $\mathfrak{vir}_-$ with basis L_{-k} ($k > 0$) with coefficients polynomials in t of degree $\leq 2j$. The constant coefficient is L_1^r and the coefficient of t^{2j} is $((2j)!)^2 L_{-r}$. More generally, the coefficient of L_{-1}^r is 1.*

Remark. We will check below that P_j is non–zero by calculating its action a specific module. In fact it can also be seen directly on a 1–dimensional module for $\mathfrak{vir}_-$. For $z\in\mathbb{C}$ define that algebra homomorphism $\psi_z : \mathfrak{vir}_- \to \mathbb{C}$ by $\psi_z(L_{-1}) = z$ and $\psi_z(L_{-k}) = 0$ for $k > 1$. Then $\psi_z(P_j) = z^{2j+1}$. Indeed we have to solve $N'w = av_j$ with $w = \sum a_k v_k$, $a_{-j} = 1$ and $N' = -F + zI$. The solution is $w = v_{-j} + zv_{-j+1} + z^2 v_{-j+2} + \cdots$ since we can check that $(zI-F)w = z^{2j+1}v_j$. Hence $\psi_j(P_j) = z^{2j+1}$. (Alternatively this can be proved using the Lie algebra endomorphism of $\mathfrak{vir}_-$ defined by $\theta(L_{-1}) = L_{-1}$ and $\theta(L_k) = 0$ for $k \leq -2$.)

Proof. The vectors ξ_k are defined inductively by $\xi_{-j} = \xi$ and for $k = -j, \ldots, j$

$$\xi_{k+1} \otimes v_k = L_{-1}\xi_k \otimes v_k - L_{-2}\xi_{k-1} \otimes tEv_{k-1} + L_{-3}t^2E^2v_{k-2} + \cdots . \qquad (*)$$

This proves uniqueness. By induction ξ_{-j+k} has the form $Q_k\xi$, with Q_k a polynomial in $\mathcal{U}_-$ of total degree k. Performing this process for the Verma module $M(1, j^2) \cong \mathcal{U}_-$, gives a uniquely determined polynomial $P_j \in \mathcal{U}_-$. Since there is a natural homomorphism of $M(c,h) \to W$ which is compatible with the equations defined by N, it is clear that $\eta = P_j\xi$.

Since the Lie algebra with basis L_{-k} ($k \geq 0$) is the semidirect product of the algebra with basis L_{-k} ($k \geq 2$) and $\mathbb{C}L_{-1}$, it makes sense to talk about the coeffcients of powers of L_{-1} in $\mathcal{U}_-$. Taking the homomorphism π sending L_{-k} to zero for $k \geq 2$ the recurrence relation becomes:

$$\pi(\xi_{k+1}) \otimes v_k = L_{-1}\pi(\xi_k) \otimes v_k,$$

so that the coefficient of L_{-1}^r is 1. Alternatively using the evaluation map σ that set $t = 0$ in the recurrence relation gives

$$\sigma(\xi_{k+1}) \otimes v_k = L_{-1}\sigma(\xi_k) \otimes v_k,$$

which yields the same result. To get the leading cofficient in t, note that by the recurrence relation ξ_{-j+k} is polynomial in t of degree at most $k-1$ for $k \leq r$. On the other hand by definition

$$\eta \otimes v_j = L_{-1}\xi_j \otimes v_j - L_{-2}\xi_{j-1} \otimes tEv_{j-1} + L_{-3}\xi_{j-2} \otimes t^2E^2v_{j-2} + \cdots + L_{-2j-1}\xi_{-j} \otimes t^{2j}E^{2j}v_{-j}.$$

Thus the highest power of t in η is t^{2j} and the coefficient is $cL_{-r}\xi$, where $cv_j = E^{2j}v_{-j}$. On the other hand $C = H^2 - H + EF$ so that

$$Ev_k = EFv_{k+1} = (j^2 + j - (k+1)^2 + k + 1)v_{k+1} = (j^2 + j - k^2 - k)v_{k+1} = (j-k)(j+k+1)v_{k+1}.$$

Thus $E^{2j}v_{-j} = ((2j)!)^2v_j$ and hence $c = ((2j)!)^2$.

Lemma C. *In the Verma module $M(c(t), h(t))$ generated by the vector ξ, there is a non–zero homogeneous polynomial P_j of total degree $(2j+1)$ in the universal enveloping algebra $\mathcal{U}_-$ of the Lie algebra with basis L_{-k} ($k > 0$) such that $P_j\xi$ is a non–zero singular vector.*

Proof. Since the map $A \mapsto A\xi$ gives an isomorphism between $\mathcal{U}_-$ and the Verma module, $P_j \neq 0$ forces $P_j\xi \neq 0$. We check that

$$L_0\eta = (j+1)^2\eta, \; L_1\eta = 0 = L_2\eta$$

Since L_1 and L_2 generate the Lie algebra with basis $L_{-k}(k > 0)$, the second identities imply that $L_k\xi_0 = 0$ for all $k > 0$, i.e. that η is a singular vector.

Let w be the vector in Lemma B with $\xi_{-j} = \xi$, $Nw = v_j \otimes \eta$ and let $P = EN$. Thus $Pw = 0$ and any solution $w' \in V \otimes M_j^2$ of $Pw' = 0$ must satisfy $Nw' = v_j \otimes \eta'$ for some $\eta' \in M_{j^2}$. On the other hand the uniqueness statement in Lemma B, if $w' = \sum v_i \otimes \xi_i'$ satisfies $\xi_{-j} = 0$ and $Pw' = 0$ then $w' = 0$.

We now show that $w' = Aw$ with $A = M, L, K$ satisfies these hypotheses. First we check that the coefficient of v_{-j} in Aw' is zero. For $A = M$, we have

$$\begin{aligned} M(v_{-j} \otimes \xi_{-j}) &= (L_0 - H - tC)(v_{-j} \otimes \xi_i) \\ &= (L_0 - H - t(j^2 + j))(v_{-j} \otimes \xi_{-j}) \\ &= ((j^2 + j)t - j + j - (j^2 + j)t)(v_{-j} \otimes \xi_{-j}) \\ &= 0. \end{aligned}$$

(In fact a similar computation shows directly that $M(v_i \otimes \xi_i) = 0$.) The result for $A = L$ and $A = K$ holds because $L_1\xi = 0 = L_2$ and Ev_i and E^2v_i are multiples of v_{i+1} and v_{i+2} respectively.

Now we check that $PAw = 0$ for the three choices of A. We first observe that, since $EH = (H - I)E$, we have $EA = A'E$ with

$$M' = L_0 - H - tC + I, \; L' = L_1 - E(t(H-2) + 1), \; K' = L_2 - tE^2(t(H - \frac{5}{2}) + \frac{7}{4}).$$

Since $[N, M] = aN$, we have $NM = MN + aN$ so that

$$ENM = EMN + aEN = M'EN + aEN.$$

Hence

$$PM = M'P + aP.$$

So that $PMw = 0$, since $Pw = 0$. By uniqueness, $Mw = 0$. Similarly $NL = LN + bEN + cM$ so that

$$ENL = ELN + bE^2N + cEM = L'EN + bE^2N + cEM.$$

Hence $PLw = 0$, since $Pw = 0$ and $Mw = 0$. By uniqueness, $Lw = 0$. Finally $NK = KN + dE^2N + eEM + fL$ so that

$$ENK = K'EN + dE^3N + eE^2M + fEL.$$

Hence $PKw = 0$, since $Nw = 0$, $Mw = 0$ and $Lw = 0$. By uniqueness, $Kw = 0$.

APPENDIX C: Proof of Feigin–Fuchs product formula using explicit formula for singular vectors.

Lemma A. *Let A be an upper triangular $n \times n$ matrix, B the lower triangular matrix with 1's below the diagonal and 0's elsewhere and suppose $A - B$ is invertible. Then the solution $v = \sum v_i e_i$ of $(A - B)v = e_1$ has $v_n = \det(A - B)^{-1}$.*

Proof. Let $X = A - B$ and $Y = X^{-1}$. Thus $v_n = y_{n1} = (-1)^{n-1} \det D / \det(X)$ where D is the $(1, n)$ minor obtained by deleting the first row and last column of X. This matrix is upper triangular with entries -1 on the diagonal, so that $\det D = (-1)^{n-1}$. Hence $v_n = (\det X)^{-1}$ as required.

Lemma B. *There is a unique vector $w = \sum w_i e_i$ with $w_n = 1$ such that $(A - B)w = ae_1$. For this solution $a = \det(A - B)$.*

Proof. The solution satisfies the recurrence relations $w_n = 1$ and for $1 \leq k \leq n$

$$w_k = \sum_{i=k}^{n} a_{ki} w_i$$

and is therefore uniquely determined. If $A - B$ is invertible the result follows immediately from Lemma A. If not then $(A - B)u = 0$ has a non–zero solution $u = \sum u_i e_i$. If $u_n \neq 0$, then, rescaling if necessary, we may assume that $u_n = 1$. But then by uniqueness $a = 0 = \det(A - B)$, as required. If $u_n = 0$, then $w' = w + u$ would satisfy $w'_n = 1$ and $(A - B)w' = e_1$, so that by uniqueness $u = 0$, a contradiction. The result follows.

Lemma C. *Let P_j be the operator giving the singular vector in the Verma module $M(1, j^2)$ with j a non–negative half integer. Let $p \geq 0$ be an integer. Then for the natural action $\ell_n = -z^{n+1} d/dz - p^2(n+1)z^n$ on $\mathbb{C}[z, z^{-1}]z^\mu$ with $c = 0$, we have $P_j z^{\mu+p^2} = az^{-d+\mu+p^2}$ where $a = (-1)^d \det(-F + (I + E)^{-1}(-\mu I - p^2 I + (2p + 1)H + jI))$.*

Remark. Only the case $p = 1$ is needed in our applications.

Proof. Let $\lambda = p^2$ and $\nu = \mu + \lambda$. We set

$$u(z) = \sum_{i=0}^{2j} a_i z^{-i+\nu} v_{-j+i}, \tag{1}$$

with $a_0 = 1$. The coefficient a is determined as the unique solution of

$$av_j z^{\nu-2j-2} = -Fu(z) - \sum_{m \geq 0} (-E)^m z^{-m} u'(z) - \sum_{m \geq 0} \lambda m (-E)^m z^{-m+1} u(z).$$

Thus

$$av_j z^{\nu-2j-2} = -Fu(z) + \sum_{m\geq 0}(-E^m)z^{-m}(\mu z^{\nu-1}v_{-j} + \sum_{i=1}^{2j}(-i+\nu)z^{-i-1+\nu}v_{-j+i}) + \lambda E(I+Ez^{-1})^{-2}u(z). \quad (2)$$

Let $w(z) = z^{-\nu}u(z)$. Thus multiplying by $z^{-\nu}$, we get

$$av_j z^{-2j-2} = -Fw(z) + \sum_{m\geq 0}(-E)^m z^{-m}(-\nu z^{-1}v_{-j} + \sum_{i=1}^{2j+}(-i+\nu)z^{-i-1}v_{-j+i} + \lambda E(I+Ez^{-1})^{-2}w(z). \quad (3)$$

Let $w = w(1) = v_{-j} + a_1 v_{-j+1} + a_2 v_{-j+2} + \cdots$ and set $z = 1$ in (3). Setting $a_0 = 1$, this yields

$$av_j = -Fw + \sum_{m\geq 0}(-E)^m \sum_{i=0}^{2j}(-i+\mu)a_i v_{-j+i} + \lambda E(I+E)^{-2}w.$$

Now

$$Hv_{-j+i} = (-j+i)v_{j-i},$$

so that

$$(-i-\mu)v_{-j+i} = (-H-j-\mu)v_{-j+i}.$$

Hence we get

$$\begin{aligned} av_j &= -Fw + \sum_{m\geq 0}(-E)^m(H+j)w + \lambda E(I+E)^{-2}w \\ &= [-F + (I+E)^{-1}(\mu I - H - jI)]w + \lambda E(I+E)^{-2}w. \end{aligned}$$

It follows from Lemma B that

$$a = \det(-F + (I+E)^{-1}(\nu I - H - jI)) + \lambda E(I+E)^{-2}). \quad (4)$$

When $\lambda = 0 = p$, this immediately gives the result for $p = 0$. When $\lambda = 1 = p$, we have $HE = EH + E$, so that $H(I+E) = (E+I)H + E$ and hence

$$(I+E)^{-1}H(I+E) = H + (I+E)^{-1}E. \quad (5)$$

Similarly $FE = EF - 2H$, so that $F(I+E) = (I+E)F - 2H$ and hence

$$(I+E)^{-1}F(I+E) = F - 2(I+E)^{-1}H. \quad (6)$$

So in this case

$$-F + (I+E)^{-1}(-\nu I + H + jI)) + E(I+E)^{-2}) = (I+E)^{-1}(-F + (I+E)^{-1}(-\nu I + 3H + jI)(I+E),$$

and hence

$$a = \det(-F + (I+E)^{-1}(-\nu I + 3H + jI).$$

It follows by induction from (5) and (6) that for $p \geq 1$

$$(I+E)^{-p}H(I+E)^p = H + p(I+E)^{-1}E,\ (I+E)^{-p}F(I+E)^p = F - 2p(I+E)^{-1}H - (p^2-p)(I+E)^{-2}E.$$

But then

$$-F + (I+E)^{-1}(H+\alpha I) + p^2(I+E)^{-2}E = (I+E)^{-p}(-F + (I+E)^{-1}(H+\alpha I))(I+E)^p.$$

Hence

$$a = \det(-F + (I+E)^{-1}(H + jI - \nu I),$$

as required.

Lemma D. *Let P_j be the operator giving the singular vector in the Verma module $M(1, j^2)$ with j a non–negative half integer. Let $p \geq 0$. Then for the natural action $\ell_n = -z^{n+1}d/dz - p^2(n+1)z^n$ on $\mathbb{C}[z, z^{-1}]z^{\mu+p^2}$ with $c = 0$, we have $Pz^{\mu+p^2} = az^{-2j-1+\mu+p^2}$ where*

$$a = (-1)^d \prod_{k\in S} (\mu - j^2 + k^2),$$

and $S = \{-j, -j+1, \cdots, j-1, j\}$ is the set of eigenvalues of H on V_j.

Proof. By Lemmas B and C, we have

$$a = \det(-F + (I+E)^{-1}(-\mu - p^2 + j + (2p+1)H)).$$

Since $\det(I+E) = 1$, it follows that

$$a = \det(-F - EF - \mu + j + H).$$

On the other hand

$$C = \frac{1}{2}(EF + FE + 2H^2) = EF - H + H^2$$

is a central operator acting on v_{-j} as $j^2 + j$. Hence

$$-EF = -j - j^2 - H + H^2,$$

so that

$$\begin{aligned} a &= \det(-F - \mu - p^2 - j - H - j^2 + H^2 + j + (2p+1)H) \\ &= \det(j^2 - 2pH - \mu - p^2 - H^2) \\ &= \det(-\mu - (H+p)^2 + j^2) \\ &= (-1)^d \prod_{k\in S} (\mu - j^2 + (k+p)^2), \end{aligned}$$

as required.

APPENDIX D: Holomorphic vector bundles and flat connections. Given a holomorphic vector bundle on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, we can restrict it to the covering by two discs $\{z : |z| < 2R\}$ and $\{z : |z| > r/2\}$ where $R > 1 > r$ which can be refined to smaller discs $\{z : |z| < R\}$ and $\{z : |z| > r\}$. We may identify each disc with a disc $D' = \{z : |z| < 1 + \delta\}$ and the smaller disc with the unit disc $D = \{z : |z| < 1\}$. Because it arises by restriction, there is a finite covering of D' by opens U_i and holomorphic maps $g_{ij} : U_i \cap U_j \to GL_n(\mathbb{C})$ with $g_{ij}g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. Taking a connection on the corresponding complex vector bundle, parallel transport along lines radiating from the origin trivialises the vector bundle. Thus there are smooth maps $h_i : U_i \to GL_n(\mathbb{C})$ such that $g_{ij} = h_i h_j^{-1}$. Thus $g_{ij} \cdot \partial_{\overline{z}} h_j = \partial_{\overline{z}} h_i$ and hence

$$h_i^{-1}\partial_{\overline{z}} h_i = h_j^{-1}\partial_{\overline{z}} h_j$$

on $U_i \cap U_j$. It follows that there is is a C^∞ map $A : D' \to M_n(\mathbb{C})$ such that $A = h_i^{-1}\partial_{\overline{z}} h_i$ on U_i. We claim that there is a smooth map $f : D' \to GL_n(\mathbb{C})$ with

$$\partial_{\overline{z}} f = -Af \qquad (*)$$

For then

$$\partial_{\overline{z}}(h_i f) = \partial_{\overline{z}}(h_i)f + h_i\partial_{\overline{z}}(f) = h_i(Af + \partial_{\overline{z}}(f)) = 0.$$

Thus $k_i = h_i f$ is holomorphic on U_i with $g_{ij} = k_i k_j^{-1}$, as required. We need the following generalisation of Dolbeault's lemma.

Lemma A. *Let $A \in C^\infty(D_1, M_s)$ and $h \in C^\infty(D_1, \mathbb{C}^s)$. Then we can find a solution $f \in C^\infty(D, \mathbb{C}^s)$ of the homogeneous equation $\partial_{\overline{z}} f = -Af$ on D. The inhomogenous equation $\partial_{\overline{z}} f = -Af + h$ on D_r is solvable if the dual equation has the* analytic continuation property*: with R fixed in $(1, 1+\delta)$, any solution of $\partial_z f = A^* f$ in $D_R = \{z : |z| < R\}$ vanishing on some open U in D_R is identically zero in D_R.*

Remark. In the example above the solutions of $\partial_{\overline{z}} f = -Af$ in $U \subset D$ correspond to holomorphic sections of the holomorphic vector bundle over U. Indeed $\xi_i(z) = h_i(z) f(z)$ satisfies $g_{ij}\xi_j = \xi_i$, so it is a section, and $\partial_{\overline{z}}\xi_i = 0$. So it is holomorphic. Conversely, if (ξ_i) is a holomorphic section, then $h_i^{-1}\xi_i$ fit together to give a function $f(z)$ which evidently satisfies $\partial_{\overline{z}} f = -Af$ in $U \subset D$. Similarly if we define dual bundle by $g'_{ij}(z) = (g_{ij}(z)^t)^{-1}$ then $h'_i(z) = (h_i(z)^t)^{-1}$ and

$$A'(z) = h_i^t (\partial_{\overline{z}} h_i^{-1})^t = -A(z)^t.$$

Similarly the corresponding quantities for the conjugate antiholomorphic bundle $g^c_{ij}(z) = (g_{ij}(z)*)^{-1}$ are $h^c_i(z) = (h_i(z)*)^{-1}$ and $A^c(z) = -A(z)^*$. Thus any solution of $\partial_z f = A^* f$ in U yields an antiholomorphic section of the antiholomorphic vector bundle over U.

Proof. Let $\psi \in C_c^\infty(D)$ be a bump function equal to 1 on D_r and 0 off D_ρ for some $r < \rho < 1$. Replacing A by ψA and h by ψh, we may assume that X lies in $C_c^\infty(D, M_s)$ and h lies in $C_c^\infty(D, \mathbb{C}^s)$ and both therefore extend to the whole of $\mathbb{C}$. To solve the equation $\partial_{\overline{z}} f = -Af + h$ on D_r, we include the disc D in a large square $F = [-R, R] \times [-R, R]$. By identifying opposite sides, doubly periodic functions on F can be identified with functions on a torus $T = \mathbb{T}^2$. The operator $\mathcal{D} = \partial_{\overline{z}} = \partial_x + i\partial_y$. Let $H_k(T)$ be the L^2 Sobolev spaces for T constructed using the Laplacian operator $\Delta = \mathcal{D}^*\mathcal{D} = \mathcal{D}\mathcal{D}^* = -\partial_x^2 - \partial_y^2$ (see **[5]**, **[18]**, **[35]** or **[38]**). The operator D defines a Fredholm operator of index 0 from $H_k(T)$ to $H_{k-1}(T)$, since it is diagonalised in the natural basis. Its kernel consists of the constant functions and its image is the orthogonal complement of the constant functions. The operator $\mathcal{D} + A$ is therefore also Fredholm of index zero from $H_k(T) \otimes M_s$ to $H_{k-1}(T) \otimes M_s$. Thus the equation

$$(\mathcal{D} + A) f = h + g \tag{1}$$

is soluble provided $(g + h, w_i)_{(k-1)} = 0$ for finitely many vectors $w_1, \ldots, w_p \in H_{k-1}(T) \otimes M_n$. But if U is an of open set $[-R, R] \times [-R, R]$ with $\overline{U} \cap \overline{D_r} = \emptyset$, then $C_c^\infty(U)$ is embedded in $H_{k-1}(T)$. We need to justify why there will be a vector g in $C_c^\infty(U)$ satisfying $(g + h, w_i)_{(k-1)} = 0$ $(i = 1, \ldots, p)$. In the homogeneous case, when $h = 0$, this is clear because the image of $C_c^\infty(U)$ contains a subspace of dimension $p + 1$.

In the inhomogeneous case, the w_i will be the smooth functions in the finite–dimensional kernel of $\mathcal{D}^* + A* = -\partial_z + A(z)^*$. By assumption their restrictions to some U with $\overline{U} \cap \overline{D_r} = \emptyset$ are linearly dependent, i.e. regarded as elements of $L^2(U, \mathbb{C}^s) \subset H_0(T)$. Since $C_c^\infty(U, \mathbb{C}^s)$ is dense in $L^2(U, \mathbb{C}^s)$, we can find a solution $g \in C_c^\infty(U, \mathbb{C}^s)$ of $(g + h, w_i) = 0$ $(i = 1, \ldots, p)$. Hence the conditions are satisfied with $k = 1$.

Corollary. *The inhomogeneous equation is always solvable in the example coming from a holomorphic vector bundle.*

Proof. Linearly independent antiholomorphic sections of the conjugate bundle stay linearly independent on any open U. Indeed in this case if w is in the kernel of $\mathcal{D}^* + A* = -\partial_z + A(z)^*$ and vanishes outside $\overline{D_r}$ then it vanishes everywhere (by antiholomorphicity inside D_R for $r < R < 1$).

Lemma B. *If $A \in C^\infty(D, M_n)$ and $D_r = \{z : |z| < r\}$ for $0 < r < 1$, then we can find $B \in C_c^\infty(D, M_n)$ such that $\partial_{\overline{z}} + A$ and $\partial_z + B$ commute on D_r.*

Proof. The commutativity condition on D_r is equivalent to

$$\partial_{\overline{z}} B = \partial_z A - \mathrm{ad}(A(z)) \cdot B,$$

which can be solved by Lemma A with $s = n^2$, $f = B$, $h = \partial_z A$ and the operator $A(z)$ given by $-\mathrm{ad}(A(z))$.

Lemma C. *If $A \in C^\infty(D, M_n)$ and $D_r = \{z : |z| < r\}$ for $0 < r < 1$, then we can find $f \in C^\infty(D, M_n)$ such that $\partial_{\overline{z}} f = -Af$ on D_r with $f(z)$ invertible on D_r.*

Proof. Taking $B(z)$ as in Lemma B, we get operators $\partial_{\overline{z}} + A$ and $\partial_z + B$ commuting on D_r. These define a flat connection so that parallel transport along any path from 0 results in an smooth map f of D_r into invertible matrices such that $f(0) = I$ and

$$\partial_{\overline{z}} f = -Af, \ \partial_z f = -Bf$$

.

Remarks. 1. Standard approximation techniques by polynomials (see [**9**]) can be applied to deduce that any holomorphic vector bundle on the unit disc is trivial.
2. The same techniques can be used to prove that a holomorphic vector bundle on an open ball B in $\mathbb{C}^n$ is trivial. In that case the we have commuting operators $\partial_{\overline{z}_i} + A_i$ and must find commuting operators $\partial_{z_i} + B_i$. This can be accomplished by successively using operators on L^2 Sobolev spaces of $\mathbb{T}^{2n}$ corresponding to the de Rham complex in the first $2k$ real variables in $\mathbb{T}^{2n}$ and the Dolbeault complex in the last $2n - 2k$ real variables (regarded as $n - k$ complex variables).

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